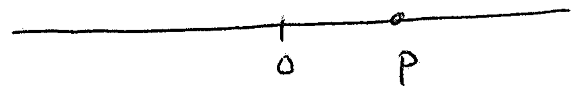
How to analyze the behavior of solutions near a critical pt of a nonlinear system:

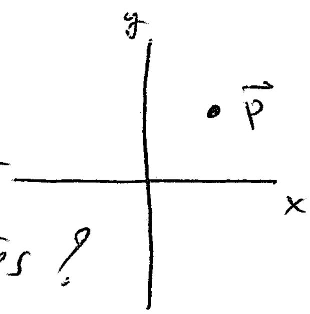
First: Understand the persistence of phase portraits under perturbations (small changes).

- Choose a pt $p \in \mathbb{R}$.



  Vary it slightly and randomly.
  Depending on your choice of p, does a slight change possibly alter its parity? Usually no.
  Yes, but only if $p = 0$.

- Choose a pt $\vec{p} \in \mathbb{R}^2$: Can a small random change alter the parity of its coordinates?



  Yes, but only if $\vec{p}$ is on one of the axes!

This is the notion of stability under small perturbations:
$\vec{p}$ is considered stable under perturbations iff neither coord. of $\vec{p}$ is 0.

~~[illegible]~~

Now let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with eigenvalues

$$r = \frac{-(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

where $p = a+d = \operatorname{tr} A$, $q = ad-bc = \det A$.

If we perturb the entries of A slightly, we perturb also p and q slightly. Hence r is perturbed.

What is the effect? For $\vec{x}' = A\vec{x}$

Sinks, saddles, sources

Type and stability persist

(a) if $r<0$ or $r>0$, perturbed will stay the same.

(b) if $r_1 \neq r_2$ This will also persist.

(c) if $r = \lambda \pm i\mu$ and $\lambda > 0$ or $\lambda < 0$, will persist.

Stars and improper nodes

Type will not persist but stability will

(d) if $r_1 = r_2$ This will not persist under random perturbations.

But the fact that $r_1 > 0$ or $r_1 < 0$ will.

Centers and such

Neither type nor stability will persist.

(e) if $r = \lambda \pm i\mu$ and $\lambda = 0$, probably will not persist. May go $\lambda > 0$ or $\lambda < 0$.

(f) if $r = 0$. May go positive or negative.

Q/ So how do solutions to $\vec{x}' = \vec{f}(\vec{x}) = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$

near a critical pt $\vec{p} \in \mathbb{R}^2$ behave?

A/ Very much like that of a linear system when the linear system is stable under perturbations:

~~Notes ① With a change in variables, $\vec{u} = \vec{x} - x$~~

Let $\vec{x}' = \vec{f}(\vec{x})$ have a critical pt $\vec{p} \in \mathbb{R}^2$.

Notes ① With a change in variables $\vec{u} = \vec{x} - \vec{p}$, the new system will ~~look~~ remain nonlinear but with a critical pt @ $\vec{u} = \vec{0}$ in $\vec{u}$-space.

② Very close to $\vec{u} = \vec{0}$, $\vec{u}' = \vec{f}(\vec{u})$ looks like a perturbed $\vec{u}' = A\vec{u}$ for some constant matrix A.

When ② is so, we call the system "almost linear" or "locally linear" at $\vec{p}$.

Def Suppose $\vec{x}' = \vec{F(\vec{x})}$ has the form

$$(\star) \qquad \vec{x}' = A\vec{x} + \vec{g(\vec{x})}$$

and $\vec{x} = \vec{0}$ is an isolated critical pt and $\det A \neq 0$. Then if $\vec{g(\vec{x})}$ has continuous partials and

$$\lim_{\vec{x} \to \vec{0}} \frac{\|\vec{g(\vec{x})}\|}{\|\vec{x}\|} = 0,$$

we say $(\star)$ is "almost linear" at $\vec{x} = \vec{0}$.

Notes ① $\vec{g(\vec{x})}$ must be small near $\vec{0}$ compared to $\vec{x}$.

② Can check $\vec{g(\vec{x})}$ component-wise:

$$\vec{g(\vec{x})} = \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \end{bmatrix}, \quad \lim_{x,y \to 0} \frac{g_i(x,y)}{r} = 0$$

where $i = 1, 2$, and $r = \sqrt{x^2 + y^2}$

ex. For F, G polynomials and $\vec{0}$ already critical, ~~this~~ ~~approx~~ finding A is easy:

Let $\left.\begin{array}{l} \dot{x} = x - x^2 - xy \\ \dot{y} = -y - y^2 + 2xy \end{array}\right\} \ \vec{x}' = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{A} \vec{x} + \underbrace{\begin{bmatrix} -x^2 - xy \\ -y^2 + 2xy \end{bmatrix}}_{\vec{g(\vec{x})}}$

Here, eigenvalues $r_1 = 1, r_2 = -1$ indicate a saddle at origin.

110.302 Lecture 30. ~~[illegible]~~

There is a much better way to characterize this:

<u>Proposition</u> A system $\vec{x}' = \vec{f}(\vec{x}) = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$ is almost linear at an isolated critical pt $\vec{x} = \vec{x}^0$ if both F, G have continuous partials up to order 2, and $Df(\vec{x}) = \begin{bmatrix} \frac{\partial F}{\partial x}\Big|_{\vec{x}^0} & \frac{\partial F}{\partial y}\Big|_{\vec{x}^0} \\ \frac{\partial G}{\partial x}\Big|_{\vec{x}^0} & \frac{\partial G}{\partial y}\Big|_{\vec{x}^0} \end{bmatrix}$

is non-degenerate.

<u>Notes</u> ① Here, $Df(\vec{x})$ is the derivative of f ~~[illegible]~~ as a vector-valued function on a domain in $\mathbb{R}^2$, called the Jacobian of f at $\vec{x}^0$. It becomes the matrix of the associated linearized system.

② Indeed, change variables to "move" $\vec{x}^0$ to the origin: let $\vec{u} = \vec{x} - \vec{x}^0 = \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$, $\vec{u}' = \vec{x}'$

So let $\vec{x}' = \vec{f}(\vec{x})$ have a critical pt at $\vec{x}^0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

Then for $u_1 = x - x_0$, $u_2 = y - y_0$, we know
$u_1' = x'$, $u_2' = y'$, and

Then

$$\vec{u}' = \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} F(x-x_0, y-y_0) \\ G(x-x_0, y-y_0) \end{bmatrix}$$

$$= \begin{bmatrix} \underbrace{\cancelto{0}{F(x_0,y_0)} + F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0)} + \eta_1(x,y) \\ \underbrace{\cancelto{0}{G(x_0,y_0)} + G_x(x_0,y_0)(x-x_0) + G_y(x_0,y_0)(y-y_0)}_{1^{st}\text{ Taylor Polynomial of } f \text{ at } \vec{x}^0} + \underbrace{\eta_2(x,y)}_{\text{Higher order stuff.}} \end{bmatrix}$$

$$= \begin{bmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} + \begin{bmatrix} \eta_1(x,y) \\ \eta_2(x,y) \end{bmatrix}$$

$$\vec{u}' = A\vec{u} + \overrightarrow{g(x)}$$

and $\overrightarrow{g(x)}$ will satisfy the smallness property since 2nd derivatives also exist.

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ex. Back to example $\dot{x} = 4-2y$, $\dot{y} = 12-3x^2$. Here there is a fixed pt at $(2,2)$. Transform $u = x-2$, $v = y-2$.

Here $\dot{u} = \dot{x}$, $\dot{v} = \dot{y}$, and

$$\dot{u} = -4-2(v+2) = -2v$$
$$\dot{v} = 12-3(u+2)^2 = -12u-3u^2$$

Then $\vec{u}' = A\vec{u} + \overrightarrow{g(\vec{u})} = \begin{bmatrix} 0 & -2 \\ -12 & 0 \end{bmatrix}\vec{u} + \begin{bmatrix} 0 \\ -3u^2 \end{bmatrix}$

So $A = \begin{bmatrix} 0 & -2 \\ -12 & 0 \end{bmatrix}$, $\vec{g(\vec{u})} = \begin{bmatrix} 0 \\ -3u^2 \end{bmatrix}$.

Since $\|\vec{g(\vec{u})}\| = \sqrt{0^2 + (-3u^2)^2} = 3u^2$, and

$\|\vec{u}\| = \sqrt{u^2+v^2} = r$, where $\begin{matrix} u = r\cos\theta \\ v = r\sin\theta \end{matrix}$, we have

$$\lim_{\vec{u}\to\vec{0}} \frac{\|\vec{g}(\vec{u})\|}{\|\vec{u}\|} = \lim_{r\to 0} \frac{3r^2\cos^2\theta}{r} = \lim_{r\to 0} 3r\cos\theta = 0$$

Hence system is almost linear at $(2,2)$.

Thm Let r_1, r_2 be the eigenvalues of the linear system $\vec{x}' = A\vec{x}$ corresponding to the almost linear $\vec{x}' = A\vec{x} + \vec{g(\vec{x})}$. Then

① if $r_1 \neq r_2$ not purely imaginary, the stability and type of the fixed pts are the same.

② if $r_1 = r_2$, the fixed pt of the nonlinear system has the same stability as that of the linear system but not necessarily the type.

③ if $r = \pm i\mu$, then neither the stability nor the type of the nonlinear critical pt can be determined by the linear system.

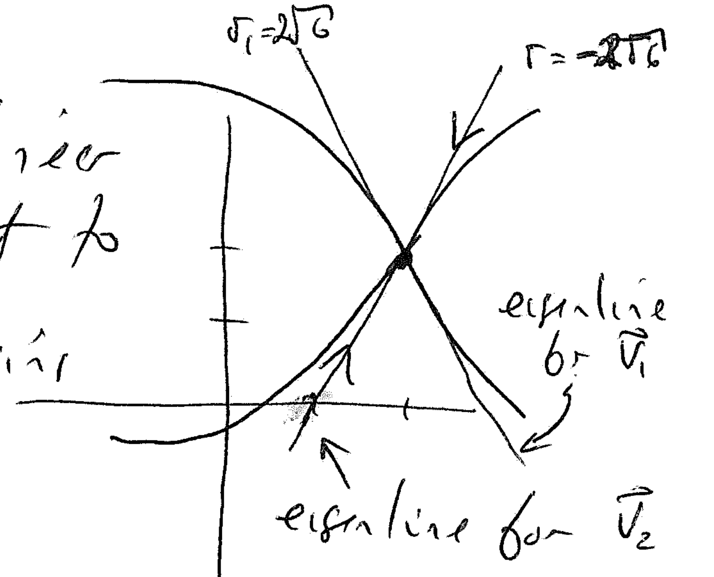
In the example, $A = \begin{bmatrix} 0 & -2 \\ -12 & 0 \end{bmatrix}$ has eigenvalues

$r = \pm 2\sqrt{6}$.

Hence the linear system has a saddle at the origin. By the Thm, then, so does the original nonlinear system at $(2,2)$.

Furthermore, for $r_1 = 2\sqrt{6}$, $r_2 = -2\sqrt{6}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ -\sqrt{6} \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ \sqrt{6} \end{bmatrix}$

To linear approximation, even the phase portraits are the same. The eigenlines of the linear system are tangent to the level set passing through $(2,2)$.



exercise: Do this for the fixed pt at $(-2,2)$.

The linear system has a center at the origin. But by the Thm we cannot tell if the original is a center or not.

Note: As long as F, G are "nice" (have cont. derivatives up to and including order 2) at a critical pt, the system is almost linear.

One calculates $A = \begin{bmatrix} F_x|_{\vec{x}^\circ} & F_y|_{\vec{x}^\circ} \\ G_x|_{\vec{x}^\circ} & G_y|_{\vec{x}^\circ} \end{bmatrix}$ and there is no need to worry about the $\vec{g}(\vec{x})$.

ex. Determine the stability of $\vec{x}^\circ = \begin{bmatrix} 2 \\ \pi \end{bmatrix}$ of $\begin{aligned} \dot{x} &= (1+x)\sin y \\ \dot{y} &= 1 - x - \cos y \end{aligned}$

Soln: Both $\begin{aligned} F(x,y) &= (1+x)\sin y \\ G(x,y) &= 1-x-\cos y \end{aligned}$ are C^∞, so system is almost linear everywhere.

Here ~~[illegible]~~ $F(2,\pi) = 0 = G(2,\pi)$, and

$$A = \begin{bmatrix} F_x(2,\pi) & F_y(2,\pi) \\ G_x(2,\pi) & G_y(2,\pi) \end{bmatrix} = \begin{bmatrix} \sin\pi & (1+2)\cos\pi \\ -1 & \sin\pi \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ -1 & 0 \end{bmatrix}$$

Eigenvalues of A satisfy $r^2 - 0r + 3 = 0$ or $r = \pm\sqrt{3}$

Hence $\vec{x}^\circ = \begin{bmatrix} 2 \\ \pi \end{bmatrix}$ is a saddle and unstable.

ex. Determine the stability of the fixed pt at $(0,0)$ of the system $\dot{x}=y$, $\dot{y}=-\gamma y-\omega^2\sin x$ for γ, ω^2 positive constants (for different values of each).

Notes: This system is almost linear everywhere since $F(x,y)=y$, $G(x,y)=-\gamma y-\omega^2\sin x$ are C^∞.

Soln: At $\begin{bmatrix}0\\0\end{bmatrix}$, $A=\begin{bmatrix}0 & 1\\ -\omega^2\cos 0 & -\gamma\end{bmatrix}=\begin{bmatrix}0 & 1\\ -\omega^2 & -\gamma\end{bmatrix}$.

with eigenvalues satisfying $r^2+\gamma r+\omega^2=0$

or $\quad r=\dfrac{-\gamma\pm\sqrt{\gamma^2-4\omega^2}}{2}$.

For different values of γ, ω^2, what can we say about the equilibrium at $\begin{bmatrix}0\\0\end{bmatrix}$?

The cases here seem to imply that the magnitude and/or sign of $\gamma^2-4\omega^2$ will determine r and hence the type and possibility of the origin.

Cases

(I) Suppose $0 < \gamma < 2|\omega|$. Then eigenvalues are complex with real part $-\frac{\gamma}{2} < 0$.

Thus the corresponding linear system has a spiral sink at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. By the thm, then, so does the nonlinear system.

(II) $\gamma = 2|\omega|$
(III) $\gamma > 2|\omega|$

$\Big\}$ Try to classify and/or understand these 2 cases.