110.302 Lecture 30. ~~[illegible]~~ V

There is a much better way to characterize this:

<u>Proposition</u> A system $\vec{x}' = \vec{f}(\vec{x}) = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$ is almost linear at an isolated critical pt $\vec{x} = \vec{x}^0$ if both F, G have continuous partials up to order 2, and
$$Df(\vec{x}) = \begin{bmatrix} \frac{\partial F}{\partial x}\Big|_{\vec{x}^0} & \frac{\partial F}{\partial y}\Big|_{\vec{x}^0} \\ \frac{\partial G}{\partial x}\Big|_{\vec{x}^0} & \frac{\partial G}{\partial y}\Big|_{\vec{x}^0} \end{bmatrix}$$
is non-degenerate.

<u>Notes</u> ① Here, $Df(\vec{x})$ is the derivative of f ~~[illegible]~~ as a vector-valued function on a domain in $\mathbb{R}^2$, called the Jacobian of f at $\vec{x}^0$. It becomes the matrix of the associated linearized system.

② Indeed, change variables to "move" $\vec{x}^0$ to the origin: let $\vec{u} = \vec{x} - \vec{x}^0 = \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$, $\vec{u}' = \vec{x}'$

So let $\vec{x}' = \vec{f}(\vec{x})$ have a critical pt at at $\vec{x}^0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

Then for $u_1 = x - x_0$, $u_2 = y - y_0$, we know

$u_1' = x'$, $u_2' = y'$, and.

Then

$$\vec{u}' = \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} F(x-x_0, y-y_0) \\ G(x-x_0, y-y_0) \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{F(x_0,y_0)} + F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0) + \eta_1(x,y) \\ \cancel{G(x_0,y_0)} + G_x(x_0,y_0)(x-x_0) + G_y(x_0,y_0)(y-y_0) + \eta_2(x,y) \end{bmatrix}$$

$\underbrace{\quad\quad}_{1^{st}\text{ Taylor Polynomial of } f \text{ at } \vec{x}^0}$ $\underbrace{\quad}_{\text{Higher order stuff.}}$

$$= \begin{bmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} + \begin{bmatrix} \eta_1(x,y) \\ \eta_2(x,y) \end{bmatrix}$$

$$\vec{u}' = A\vec{u} + \overrightarrow{g(x)}$$

and $\overrightarrow{g(x)}$ will satisfy the smallness property since 2^{nd} derivatives also exist.

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ex. Back to example $\dot{x} = 4-2y$, $\dot{y} = 12-3x^2$. Here there is a fixed pt at $(2,2)$. Transform $u = x-2$, $v = y-2$.

Here $\dot{u} = \dot{x}$, $\dot{v} = \dot{y}$, and

$$\dot{u} = -4-2(v+2) = -2v$$
$$\dot{v} = 12-3(u+2)^2 = -12u-3u^2$$

Then $\vec{u}' = A\vec{u} + \overrightarrow{g(\vec{u})} = \begin{bmatrix} 0 & -2 \\ -12 & 0 \end{bmatrix}\vec{u} + \begin{bmatrix} 0 \\ -3u^2 \end{bmatrix}$

So $A = \begin{bmatrix} 0 & -2 \\ -12 & 0 \end{bmatrix}$, $\vec{g(\vec{u})} = \begin{bmatrix} 0 \\ -3u^2 \end{bmatrix}$.

Since $\|\vec{g(\vec{u})}\| = \sqrt{0^2 + (-3u^2)^2} = 3u^2$, and

$\|\vec{u}\| = \sqrt{u^2+v^2} = r$, where $\begin{matrix} u = r\cos\theta \\ v = r\sin\theta \end{matrix}$, we have

$$\lim_{\vec{u}\to\vec{0}} \frac{\|\vec{g}(\vec{u})\|}{\|\vec{u}\|} = \lim_{r\to 0} \frac{3r^2\cos^2\theta}{r} = \lim_{r\to 0} 3r\cos\theta = 0$$

Hence system is almost linear at $(2,2)$.

__Thm__ Let r_1, r_2 be the eigenvalues of the linear system $\vec{x}' = A\vec{x}$ corresponding to the almost linear $\vec{x}' = A\vec{x} + \vec{g(\vec{x})}$. Then

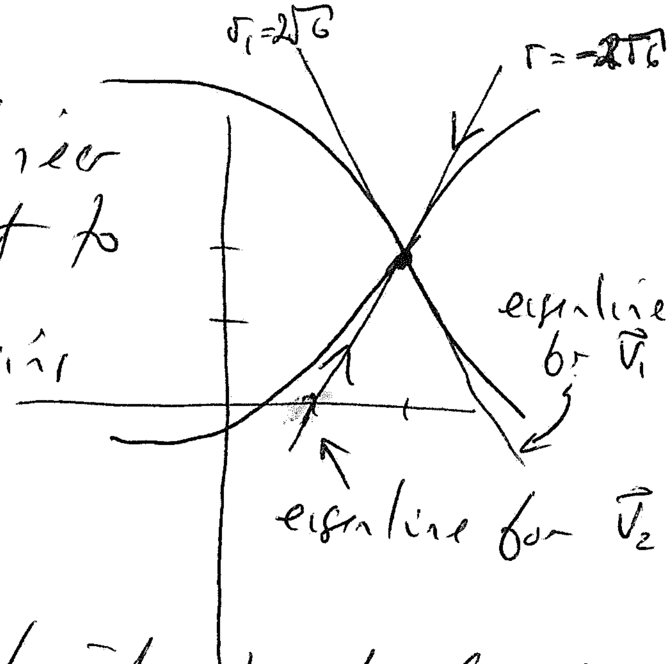
1. if $r_1 \neq r_2$ not purely imaginary, the stability and type of the fixed pts are the same.
2. if $r_1 = r_2$, the fixed pt of the nonlinear system has the same stability as that of the liner system but not necessarily the type.
3. if $r = \pm i\mu$, then neither the stability nor the type of the nonlinear critical pt can be determined by the liner system.

In the example, $A = \begin{bmatrix} 0 & -2 \\ -12 & 0 \end{bmatrix}$ has eigenvalues $r = \pm 2\sqrt{6}$.

Hence the linear system has a saddle at the origin. By the Thm, then, so does the original nonlinear system at $(2,2)$.

Furthermore, for $r_1 = 2\sqrt{6}$, $r_2 = -2\sqrt{6}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ -\sqrt{6} \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ \sqrt{6} \end{bmatrix}$

To linear approximation, even the phase portraits are the same. The eigenlines of the linear system are tangent to the level set passing through $(2,2)$.



exercise: Do this for the fixed pt at $(-2,2)$.
The linear system has a center at the origin. But by the Thm we cannot tell if the original is a center or not.

Note: As long as F, G are "nice" (have cont. derivatives up to and including order 2) at a critical pt, the system is almost linear. One calculates $A = \begin{bmatrix} F_x|_{\vec{x}^0} & F_y|_{\vec{x}^0} \\ G_x|_{\vec{x}^0} & G_y|_{\vec{x}^0} \end{bmatrix}$ and there is no need to worry about the $\vec{g}(\vec{x})$.

ex. Determine the stability of $\vec{x}^0 = \begin{bmatrix} 2 \\ \pi \end{bmatrix}$ of $\begin{aligned} \dot{x} &= (1+x)\sin y \\ \dot{y} &= 1 - x - \cos y \end{aligned}$

<u>Soln</u>: Both $\begin{aligned} F(x,y) &= (1+x)\sin y \\ G(x,y) &= 1-x-\cos y \end{aligned}$ are C^∞, so system is almost linear everywhere.

Here ~~[illegible]~~ $F(2,\pi) = 0 = G(2,\pi)$, and

$$A = \begin{bmatrix} F_x(2,\pi) & F_y(2,\pi) \\ G_x(2,\pi) & G_y(2,\pi) \end{bmatrix} = \begin{bmatrix} \sin\pi & (1+2)\cos\pi \\ -1 & \sin\pi \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ -1 & 0 \end{bmatrix}$$

Eigenvalues of A satisfy $r^2 - 0r + 3 = 0$ or $r = \pm\sqrt{3}$

Hence $\vec{x}^0 = \begin{bmatrix} 2 \\ \pi \end{bmatrix}$ is a saddle and unstable.

ex. Determine the stability of the fixed pt at $(0,0)$ of the system $\dot{x} = y$, $\dot{y} = -\gamma y - \omega^2 \sin x$ for γ, ω^2 positive constants (for different values of each).

Notes: This system is almost linear everywhere since $F(x,y) = y$, $G(x,y) = -\gamma y - \omega^2 \sin x$ are C^∞.

Soln: At $\begin{bmatrix}0\\0\end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ -\omega^2 \cos 0 & -\gamma \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{bmatrix}$.

with eigenvalues satisfying $r^2 + \gamma r + \omega^2 = 0$

or $\quad r = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4\omega^2}}{2}$.

For different values of γ, ω^2, what can we say about the equilibrium at $\begin{bmatrix}0\\0\end{bmatrix}$?

The cases here seem to imply that the magnitude and/or sign of $\gamma^2 - 4\omega^2$ will determine r and hence the type and possibility of the origin.

Cases

(I) Suppose $0 < \gamma < 2|\omega|$. Then eigenvalues are complex with real part $-\frac{\gamma}{2} < 0$.

Thus the corresponding linear system has a spiral sink at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. By the thm, then, so does the nonlinear system.

(II) $\gamma = 2|\omega|$
(III) $\gamma > 2|\omega|$

$\Big\}$ Try to classify and/or understand these 2 cases.