110.302 Lecture 31:

More nonlinear behavior

Near a center, all trajectories are ~~closed~~ periodic.

(curves $\vec{x}(t)$ where there exists a positive number $T>0$ where $\vec{x}(t+T)=\vec{x}(t)$ for all $t\in\mathbb{R}$.)

We call these curves closed.

Is it possible to have only one such curve in the phase plane?

ex.
$$\dot{x} = x - y - x(x^2+y^2)$$
$$\dot{y} = x + y - y(x^2+y^2)$$

Fixed pts? $(0,0)$ is one....

This system is almost linear @ $(0,0)$, and @ $(0,0)$, associated linear system has $A=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$ with eigenvalues $r = 1\pm i$.

The origin is a spiral source.

What else is going on? Notice a pattern in x^2+y^2?
What if we switched coordinates to polar?

If $$\left.\begin{aligned} x(t) &= r(t)\cos\theta(t) \\ y(t) &= r(t)\sin\theta(t) \end{aligned}\right\} \qquad \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

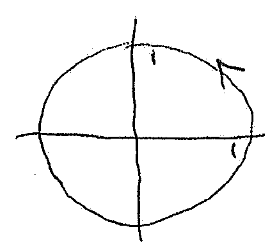
Then $x^2+y^2=r^2$ and $\theta=\arctan\left(\frac{y}{x}\right)$

exercise: Show transformed system is

$$\dot{r}=r(1-r^2)$$
$$\dot{\theta}=1.$$

This system is uncoupled. Without solving, fixed pts?

- The origin $r=0$ $\theta=$anything is fixed.
- But so is $r(t)=1$, $\theta(t)=t$.

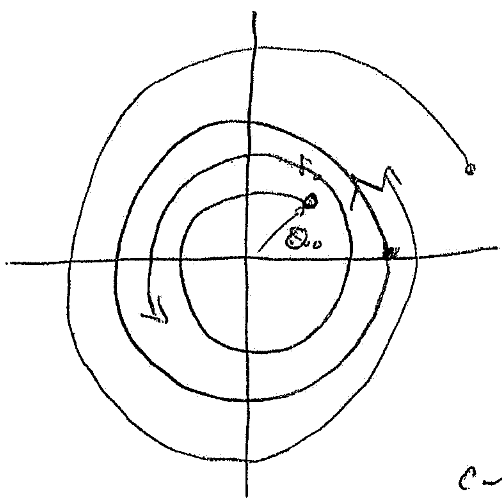


This orbit (trajectory) is closed, and is called a cycle

- What happens if we start
  1. On the circle
  2. inside the circle
  3. Outside the circle

- System is autonomous, and $f(r,\theta)$ has derivatives of all orders. $\Longrightarrow$ Solutions are unique (trajectories cannot cross).

- Suppose we start @ $r_0 = r(t_0)$, $\theta_0 = \theta(t_0)$, where
  - $r_0 < 1$ (inside the circle).
    - Here $\dot{\theta} = 1$, $\dot{r} > 0$ always.
  - $r_0 > 1$ (outside the circle)
    - Here $\dot{\theta} = 1$, $\dot{r} < 0$ always

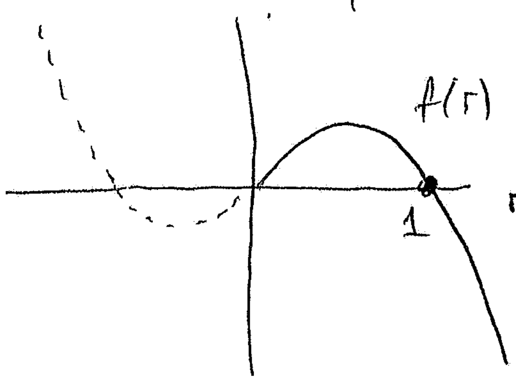


- It turns out that <u>ALL</u> trajectories tend toward the cycle @ $r = 1$.

- $r(t) \equiv 1$ is called a <u>limit</u> cycle

and since for any $r_0 > 0$, $\lim_{t\to\infty} r(t) = 1$, $r(t) \equiv 1$ is asymptotically stable.

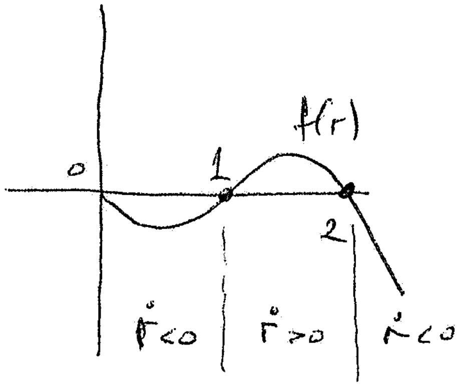
- Another way to see this: $\dot{r} = r(1-r^2) = f(r)$
$\dot{\theta} = 1$

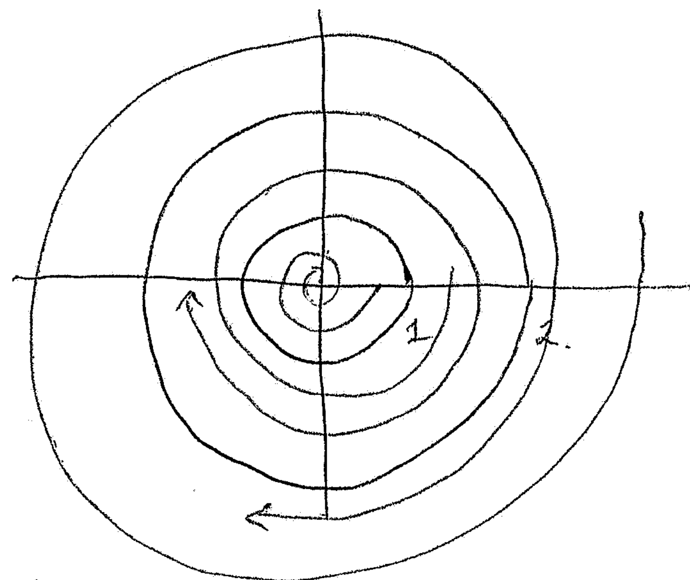


Here all trajectories are either closed (origin and $r \equiv 1$) or limit to a closed trajectory.

ex Can you draw the phase portrait for

$$\dot{r} = r(1-r)(r-2) = f(r)$$
$$\dot{\theta} = -1$$

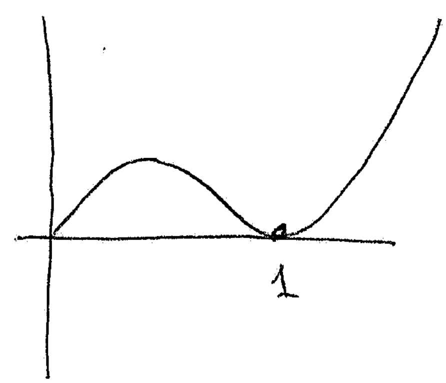




Conclusion: • The origin is a spiral sink. The limit cycle $r(t) \equiv 1$ is unstable and the limit cycle $r(t) \equiv 2$ is asympt. stable.

ex. How about $\dot{r} = r(1-r)^2$

Here $r(t) \equiv 1$ is called semi-stable. why?



Q: In any autonomous ODE system with unique solutions, what are the options for for the long term behavior of trajectories:

① $\longrightarrow \infty$

② $\longrightarrow$ fixed pt (equilibrium)

③ $\longrightarrow$ closed trajectory (cycle)

④ $\longrightarrow$?

In higher dimensions there is a 4th option (strange attractor, eg.), but not in 2D.

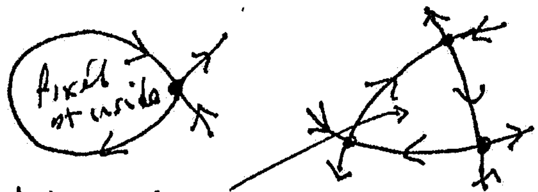
<u>Qualitative</u> <u>Existence</u> <u>Thms</u>

Some results involving system $\dot{x} = F(x,y)$, $\dot{y} = G(x,y)$.

<u>Thm 1</u> Let F, G have continuous partials on a domain $D \subset \mathbb{R}^2$. Then ① Any closed nontrivial trajectory must contain at least 1 critical pt.
② If there is only 1, then it cannot be a saddle.

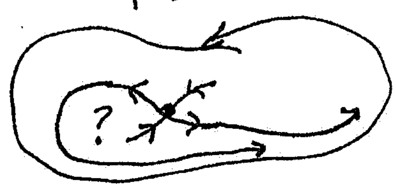
<u>Contrapositive</u>: If there does not exist a fixed pt, then there cannot exist a closed trajectory!

Notes Ⓐ Also works if a finite number of whole trajectories together make a closed curve:



must be a fixed pt inside

Ⓑ Try to visually show part ② is wrong.



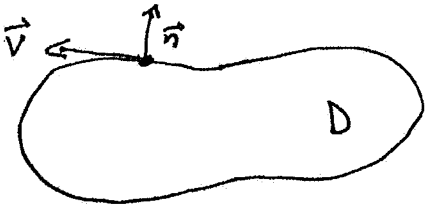
Thm 2 Let F, G have continuous partials on a simply connected domain D (no holes inside). If $F_x + G_y$ has the same sign throughout D, then there does not exist a $\underbrace{\text{closed nontrivial}}_{\text{cycle}}$ trajectory completely in D.

ex. $\left.\begin{array}{l}\dot{x} = x \\ \dot{y} = y\end{array}\right\}$ Here $F_x = G_y = 1$, $D = \mathbb{R}^2$

Note: In Calc III, any ODE $\begin{array}{l}\dot{x} = F(x,y) \\ \dot{y} = G(x,y)\end{array}$ defines a vector field on $\mathbb{R}^2$

$\vec{V} = F\vec{i} + G\vec{j}$. Here for any closed bounded domain D,

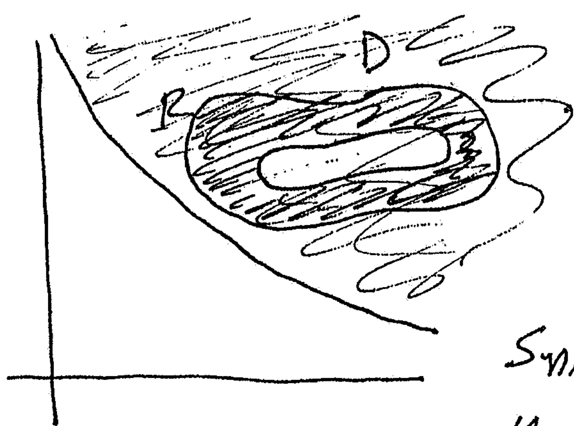
we have $\displaystyle\oint_{\partial D} \vec{V} \cdot \vec{n}\, ds = \iint_D (F_x + G_y)\, dA$



If $F_x + G_y$ as in thm, then RHS $\neq 0$. But if ∂D is a closed trajectory, then LHS $= 0$.

Hence one cannot find any D w/ cycle as boundary.

Thm 3 Poincare - Bendixson.



Let F, G have continuous partials on $D \subset \mathbb{R}^2$ (may be with holes), D_1 be a bounded subdomain and R the closure of D_1.

Suppose R contains no critical pts. If there exists a t_0 where for all $t \geq t_0$, a solution $x = \varphi(t)$, $y = \psi(t)$ enters R and never leaves, then either $x = \varphi(t)$, $y = \psi(t)$ is a closed trajectory or is asymptotic to a closed trajectory.

Conclusion: $\exists$ a closed trajectory in R.

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ex. Van der Pol's equation $u'' - \mu(1-u^2)u' + u = 0$

(kinda looks like the pendulum with μ as damping)

Here μ is a non-negative constant. When $\mu = 0$ all solutions are periodic. ($u'' + u = 0$).

For $\mu > 0$, it is not clear what the dynamics are.

Switch to a system

$$\dot{x} = y$$
$$\dot{y} = -x + \mu(1-x^2)y.$$

Here, it is obvious the origin is fixed.

In polar coordinates, $\dot{r} = \mu(1 - r^2\cos^2\theta) r \sin^2\theta$

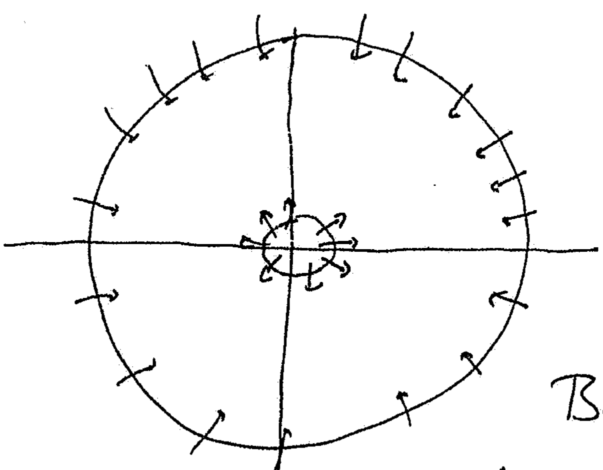
$$\dot{\theta} = -1 - \underbrace{\mu(r^2\cos^2\theta - 1)\sin\theta\cos\theta}_{\text{usually much smaller than } 1.}$$

exercise: Show origin is ~~an~~ source for $\mu > 0$ (it is a spiral for $\mu < 2$ and a node for $\mu > 2$).

exercise: Show for $r >> 0$, $\dot{r} < 0$

exercise: Show the origin is the only fixed pt.

Consider the annulus given by the region between $r = \varepsilon > 0$ for small ε, and a large $\frac{1}{\varepsilon}$ satisfying exercise 2.



Call this R. On inner ring $\dot{r} > 0$, on outer ring $\dot{r} < 0$, so any trajectory that starts inside R stays inside R for ever.

By PB-thm, there must exist a closed trajectory in R.

Thus there is a *periodic solution*