More nonlinear behavior

Near a center, all trajectories are closed. (curves will be periodic.)

A solution is periodic if there exists a positive number $T > 0$ such that

$$\bar{x}(t+T) = \bar{x}(t) \text{ for all } t \in \mathbb{R}.$$ 

We call such curves closed.

Is it possible to have only one such curve in the phase plane?

Example:

$$\begin{cases} \dot{x} = x - y - x(x^2 + y^2) \\ \dot{y} = x + y - y(x^2 + y^2) \end{cases}$$

Fixed at? $(0,0)$ is one...

This system is almost linear at $(0,0)$, and at $(x_0, y_0)$, associated linear system has $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

with eigenvalues $\lambda = 1 \pm i$.

The origin is a spiral source.
What else is going on? Notice a pattern in $x^2 + y^2$?

What if we switched coordinates to polar?

\[
\begin{align*}
16 & \quad x(t) = r(t) \cos \theta(t) \quad x = r \cos \theta \\
& \quad y(t) = r(t) \sin \theta(t) \quad y = r \sin \theta
\end{align*}
\]

Note $x^2 + y^2 = r^2$ and $\theta = \arctan \left( \frac{y}{x} \right)$

Exercise: Show transformed system is

\[
\begin{align*}
\dot{r} &= r(1 - r^2) \\
\dot{\theta} &= 1
\end{align*}
\]

This system is uncoupled. Without solving, fix $\phi$ at?

- The origin $r = 0, \theta = \text{ arbitrary} \quad \text{fixed}$
- But $r = 0 \implies M = 1, \theta(t)$

This orbit (trajectory) is closed, and is called a **circle**.

What happens if we shift
- (a) On the circle
- (b) Inside the circle
- (c) Outside the circle
- System is autonomous, and \( f(r, \theta) \) has derivatives of all orders. \( \Rightarrow \) Solution of unique (trajectories cannot cross).

- Suppose \( r_0 \) is a solution of \( \Gamma_0 \), \( \theta_0 = \theta(t_0) \), where
  - \( r_0 < 1 \) (inside the circle).
    - Here \( \dot{\theta} = 1, \dot{r} > 0 \) always.
  - \( r_0 > 1 \) (outside the circle).
    - Here \( \dot{\theta} = 1, \dot{r} < 0 \) always.

It turns out that **all trajectories** tend toward the cycle \( \Gamma = 1 \).

- \( \Gamma = 1 \) is called a limit cycle and since for any \( r_0 > 0 \), \( t \rightarrow \infty, r(t) = 1, \quad \Gamma(t) = 1 \) is asymptotically stable.

Another way to see this:
\[ \dot{r} = f(1 - r^2) = f(r) \]
\[ \dot{\theta} = 1 \]

Here all trajectories are either closed (origin and \( r=1 \)) or limit to a closed trajectory.
Consider drawing the phase portrait for
\[ \dot{r} = r(1-r)(r-2) = f(r) \]
\[ \dot{\theta} = -1 \]

Conclusion: The origin is a spiral sink. The limit cycle \( r = 1 \) is unstable and the limit cycle \( r = 2 \) is asymptotically stable.

ex. How about \( \dot{r} = r(1-r)^2 \)

Here \( r = 1 \) is called semi-stable. Why?
Q: In any autonomous ODE system with unique solutions, what are the options for the long-term behavior of trajectories:

1. \( \rightarrow \infty \)
2. \( \rightarrow \text{fixed point (equilibrium)} \)
3. \( \rightarrow \text{closed trajectory (cycle)} \)
4. \( \rightarrow ? \)

In higher dimensions there is a 5th option (strange attractor e.g.), but not in 2D.

**Qualitative Existence Theorems**

Some results involving system \( x = F(x, t), \ y = G(x, t) \).

**Theorem**: Let \( F, G \) have continuous partials on a domain \( \mathbb{R}^2 \). Then

1. Any closed nontrivial trajectory must contain at least 1 critical point.

2. If there is only 1, then it cannot be a saddle.

**Contrapositive**: If there does not exist a fixed pt, then there cannot exist a closed trajectory!
Notes:

(4) Also works if a finite number of whole trajectories together make a closed curve.

(5) Try to visually show that (2) is wrong.

Thm 2: Let \( \mathbf{F} \) have continuous partials on a simply connected domain \( D \) (no holes inside). If \( \mathbf{F} \) has the same sign throughout \( D \), then there does not exist a closed nontrivial trajectory completely in \( D \).

Example:

\[ \begin{align*}
\dot{x} &= x^2 \\
\dot{y} &= y 
\end{align*} \]

Here \( \mathbf{F} = (x, y) \), \( D = \mathbb{R}^2 \).

Note: In Case III, can ODE \( \dot{x} = F(x, y) \) defined a vector field on \( \mathbb{R}^2 \).

\( \mathbf{V} = F \dot{x} + G \dot{y} \). Here \( \mathbf{V} \) an closed bounded domain \( D \),

we have

\[ \oint_D \mathbf{V} \cdot \mathbf{n} \, ds = \iint_D (F_x + G_y) \, dA \]

16. For this as in Thm, then RHS \( \neq 0 \). But if \( \partial D \) is a closed trajectory, then LHS \( = 0 \).

Hence one cannot find an \( \mathbf{F} \) on \( D \) w/ cycle as boundary.
Theorem 3: Poincaré–Bendixson.

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field defined on a bounded region $D$. Suppose $R$ is a closed curve in $D$.

**Main Result:**
- Suppose $R$ contains no critical points.
- If there exists a solution $x(t), y(t)$ that enters $R$ and never leaves, then either $x(t), y(t)$ is a closed trajectory or is asymptotic to a closed trajectory.

**Conclusion:** If a closed trajectory in $R$.

**Example:** Van der Pol's equation $\dddot{x} - \mu(1-x^2)\ddot{x} + x = 0$.
- (Kind of looks like the pendulum with $\mu$ as damping.)
- Here $\mu$ is a non-negative constant. When $\mu = 0$, all solutions are periodic. ($\dddot{x} + x = 0$).

For $\mu > 0$, it is not clear what the dynamics are.

Switch to a system:

\[ \begin{align*}
    \dot{x} &= y \\
    \dot{y} &= -x + \mu(1-x^2)y.
\end{align*} \]

Here, it is obvious the origin is fixed.
In polar coordinates, 
\[ r = \mu (1 - r^2 \cos^2 \theta) \sin^2 \theta \]
\[ \dot{\theta} = -1 - \frac{\mu (r^2 \cos^2 \theta - 1) \sin \theta \cos \theta}{\text{usually much smaller than 1}} \]

Exercises: Show origin is an **source** for \( \mu > 0 \) (it is a spiral for \( \mu < 2 \) and a node for \( \mu > 2 \)).

**Exercise:** Show for \( r > 0 \), \( \dot{r} < 0 \)

**Exercise:** Show the origin is the only fixed pt.

Consider the annulus given by the region between \( r = \epsilon > 0 \)
for small \( \epsilon \), and a large \( \epsilon \) satisfying exercise 2.

Call this \( R \). On inner ring \( \dot{r} > 0 \),
on outer ring \( \dot{r} < 0 \), so can trajectory that starts inside \( R \)
stays inside \( R \) for ever.

By Poincaré-Bendixson, there must exist a closed trajectory \( -R \).

Thus there is a periodic solution.