

Math 110,302 Lecture 38: ~~XXXXXXXXXX~~

Recall the improper integral

$$\int_a^{\infty} f(t) dt = \lim_{A \rightarrow \infty} \int_a^A f(t) dt$$

Has LHS converges if limit exists, diverges if not

ex. $\int_a^{\infty} 4 dt$ div $\forall a \in \mathbb{R}$

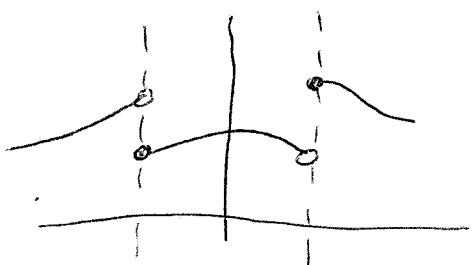
ex. $\int_0^{\infty} e^{-kt} dt = \lim_{A \rightarrow \infty} \int_0^A e^{-kt} dt = \lim_{A \rightarrow \infty} \left[\frac{e^{-kA}}{-k} + \frac{1}{k} \right]$
 $= \frac{1}{k}$ converges when $k > 0$.

Recall $f(t)$ is piecewise continuous on $a \leq t \leq \beta$

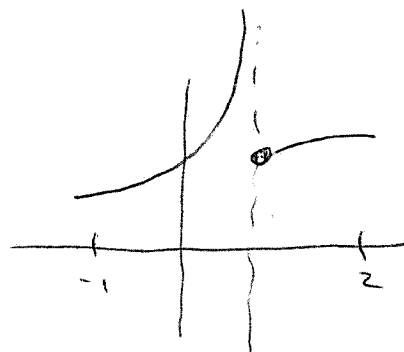
if \exists partition $a = t_0 < t_1 < \dots < t_{n-1} < t_n = \beta$

where

- (1) $f(t)$ is continuous on each (t_{i-1}, t_i)
- (2) Both one-sided limits exist on each (t_{i-1}, t_i) .



piecewise cont



not piecewise continuous on $(-1, 2)$.

Def A transform (integral) is a relation of the form

$$F(s) = \int_{\alpha}^{\beta} K(s, t) A(t) dt$$

Here $K(s, t)$ is called the kernel of the transform and $\alpha \geq -\infty, \beta \leq \infty$. $F(s)$ becomes the transformed $A(t)$.

Reason to transform: Certain problems become easier to solve....

ex. (Not a transform, but...)

Calculate $y'(x)$, for $y(x) = \frac{x^{3/4} \sqrt{x^2+1}}{(3x+2)^5}, x > 0$.

Idea: Since $y > 0$, for $x > 0$, instead 1st change problem:

Logarithmic Differentiation: Given

$$\ln y = \frac{3}{4} \ln x + \frac{1}{2} \ln(x^2+1) - 5 \ln(3x+2),$$

$$\frac{d}{dx} [\ln y] = \frac{y'}{y} = \frac{3}{4x} + \frac{2x}{2(x^2+1)} - \frac{15}{3x+2}$$

$$\Rightarrow y' = y \left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2} \right)$$

Transforms have a similar effect

Def Laplace Transform

Let $f(t)$ be defined for $t \geq 0$ with some "added properties".

$$\Rightarrow \mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

is the Laplace Transform of $f(t)$ whenever the integral converges.

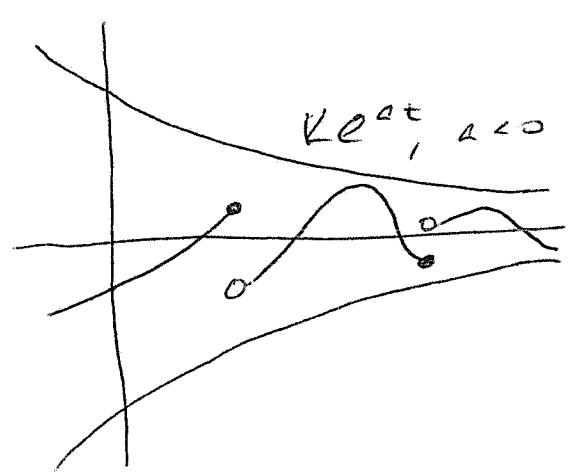
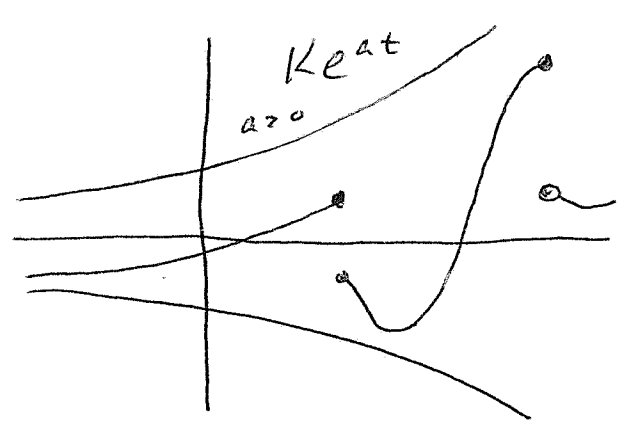
Notes

① In general, s is a complex variable, but for now, keep it real. (s is called complex frequency).

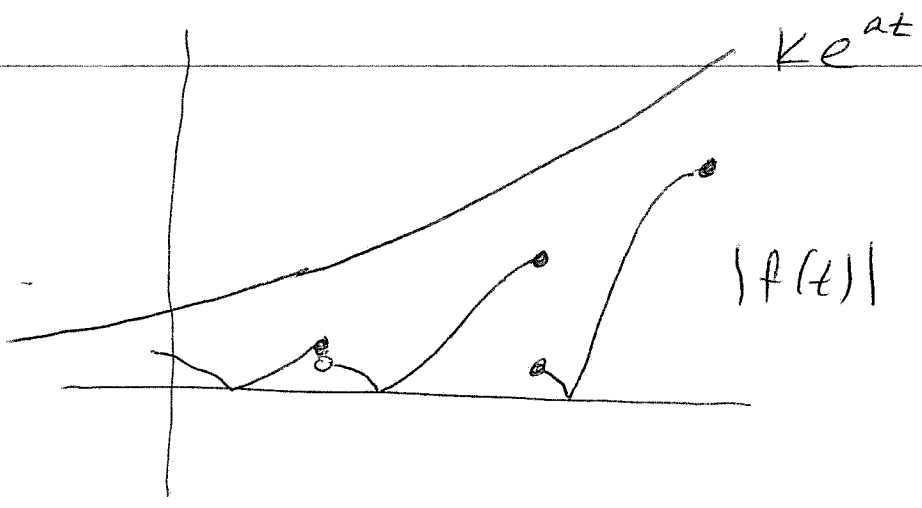
② "added properties":

Ⓐ f is piecewise cont. on $[0, A]$ for $A > 0$

Ⓑ $|f(t)| \leq Ke^{at}$, for $t \geq M$, where $K > 0, M > 0, a$ are all real.



For (b), we only require that f does not increase faster than exponentially.



Thm If $f(t)$ is defined so that (a) & (b) hold, then

$$\mathcal{L}\{f(t)\} = F(s)$$

exists for $s > a$.

pt. Need to show $\int_0^\infty e^{-st} f(t) dt$ converges.

Here $\int_0^\infty e^{-st} f(t) dt = \underbrace{\int_0^M e^{-st} f(t) dt}_{\substack{\text{converges} \\ \text{by (a)}}} + \int_M^\infty e^{-st} f(t) dt$

$$\dots \leq \int_M^\infty e^{-st} f(t) dt \leq \left| \int_M^\infty e^{-st} f(t) dt \right| \leq \int_M^\infty e^{-st} |f(t)| dt = \int_M^\infty e^{-st} |f(t)| dt$$

$$\leq \int_M^\infty e^{-st} k e^{at} dt = k \int_M^\infty e^{(a-s)t} dt$$

which will converge as $s > a$.

examples of Laplace Transforms

Ⓘ Let $f(t) \equiv 1, t > 0$:

$$\begin{aligned} \mathcal{L}\{1\} &= \int_0^{\infty} e^{-st} dt = \lim_{A \rightarrow \infty} \int_0^A e^{-st} dt \\ &= \lim_{A \rightarrow \infty} \left. \frac{e^{-st}}{-s} \right|_0^A = \lim_{A \rightarrow \infty} \frac{e^{-sA} - 1}{-s} = \frac{1}{s}, s > 0 \end{aligned}$$

So for $f(t) \equiv 1, t > 0$, $F(s) = \frac{1}{s}, s > 0$.

Q: How does this change for $f(t) = k, t > 0, k \in \mathbb{R}$.

Ⓜ Let $f(t) = e^{at}, t > 0$:

$$\mathcal{L}\{e^{at}\} = \int_0^{\infty} e^{-st} e^{at} dt = \int_0^{\infty} e^{-(s-a)t} dt = \frac{1}{s-a}, s > a$$

NOTES ① Domains are important: Ⓜ only works for $s > a$!

② Notice how Ⓜ is Ⓘ when a is chosen to be 0.

Ⓝ Header stuff

$$\begin{aligned} \mathcal{L}\{\sin at\} &= F(s) = \int_0^{\infty} e^{-st} \sin at dt \\ &= \lim_{A \rightarrow \infty} \int_0^A e^{-st} \sin at dt \end{aligned}$$

By integration by parts

$$F(s) = \frac{1}{2} - \frac{s}{2} \int_0^{\infty} e^{-st} \cos at dt$$

Another integration by parts

$$\mathcal{L}\{\sin at\} = F(s) = \frac{1}{a} - \frac{s^2}{a^2} F(s)$$

or

$$F(s) = \frac{a}{s^2 + a^2}, \quad s > 0$$

Q: Why is LT useful?

First, $\mathcal{L}\{f(t)\}$ is an operator, and it is linear (integral operators are linear).

$$\begin{aligned} \mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} &= \int_0^{\infty} e^{-st} (c_1 f_1(t) + c_2 f_2(t)) dt \\ &= c_1 \int_0^{\infty} e^{-st} f_1(t) dt + c_2 \int_0^{\infty} e^{-st} f_2(t) dt \\ &= c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}. \end{aligned}$$

Second It behaves well with derivatives

Thm Suppose $f(t)$ is cont on $t \in [0, \infty)$ and $f'(t)$ piecewise cont. there, and $|f'(t)| \leq Ke^{at}$, $t \geq M$

$$\Rightarrow \mathcal{L}\{f'(t)\} = s \mathcal{L}\{f(t)\} - f(0)$$

on $s > a$.

Integration by
Parts

Quick proof

pt. Given $f(t)$, $t \geq 0$, continuous,

$$\mathcal{L}\{f'(t)\} = \int_0^{\infty} e^{-st} f'(t) dt$$

$$= \lim_{A \rightarrow \infty} \int_0^A e^{-st} f'(t) dt$$

$$\stackrel{\text{Int parts}}{=} \lim_{A \rightarrow \infty} \left(e^{-st} f(t) \Big|_0^A - \int_0^A (-s) e^{-st} f(t) dt \right)$$

$$= -f(0) + s \int_0^{\infty} e^{-st} f(t) dt$$

$$= s \mathcal{L}\{f(t)\} - f(0) \quad \square$$

Note: Can also recursively apply this formula to get

$$\mathcal{L}\{f''(t)\} = s \mathcal{L}\{f'(t)\} - f'(0)$$

$$= s(s \mathcal{L}\{f(t)\} - f(0)) - f'(0)$$

$$= s^2 \mathcal{L}\{f(t)\} - s f(0) - f'(0)$$

up to

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

provided all der $f(t), \dots, f^{(n-1)}(t)$ are cont on $t \geq 0$,
 $f^{(i)}(t)$ is piecewise cont. and all derivatives satisfy

$$|f^{(i)}(t)| \leq K e^{ct}, \quad t \geq M$$

for $i = 0, \dots, n-1$.