Example: Solve the IVP 
\[ y'' - 3y' - 4y = 0, \quad y(0) = 1 \]
\[ y'(0) = 2. \]

Solution: Since LT is linear, we can transform the entire equation:

\[ L\{ y'' - 3y' - 4y \} = 0 \]
\[ L\{ y'' \} - 3L\{ y' \} - 4L\{ y \} = 0 \]

\[ \begin{align*}
    s^2L\{ y \} - sy(0) - y'(0) - 3[sL\{ y \} - y(0)] - 4L\{ y \} &= 0 \\
    s^2L\{ y \} - 5 - 2 - 3sL\{ y \} + 3 - 4L\{ y \} &= 0 \\
    s^2L\{ y \} - 3L\{ y \} &= 0 \\
    \implies L\{ y \} &= \frac{3}{s^2 - 3s - 4} = \frac{3}{(s-4)(s+1)} = \frac{\frac{3}{5}}{s-4} + \frac{\frac{1}{5}}{s+1} \\
    \end{align*} \]

Solve by partial fraction decomposition to get

\[ L\{ y \} = \frac{3}{s} \left( \frac{\frac{1}{5}}{s-4} - \frac{\frac{1}{5}}{s+1} \right) \]

What functions have these LTs?

\[ L\{ e^{4t} \} = \frac{1}{s-4}, \quad L\{ e^{-t} \} = \frac{1}{s+1} \]

\[ \implies y(t) = \frac{3}{5} e^{4t} + \frac{1}{5} e^{-t} \]

Now solve it the other way in your notes!
Advantages?

1. Works equally well with higher order ODEs.
2. Turns an ODE into an algebraic equation.
3. Many basic functions have well-known LTs.

4. Incorporates both non-homogeneous terms and initial data into the calculation.

ex. Solve $x'' + 3x' + 2x = 12e^{2t}$, $x(0) = 1$, $x'(0) = -1$.

Here $\mathcal{L}\{x'' + 3x' + 2x = 12e^{2t}\}$

$\mathcal{L}\{x''\} + 3\mathcal{L}\{x'\} + 2\mathcal{L}\{x\} = 12\mathcal{L}\{e^{2t}\}$

Finish this calculation:

The solution is $x(t) = e^{2t} + 3e^{-2t} - 3e^{-t}$

ex. Solve $y^{(4)} - y = 0$, $y(0) = y''(0) = y'''(0) = 0$, $y'(0) = 1$.

Here $\mathcal{L}\{y^{(4)} - y\} \Rightarrow 0$ becomes

$5^4\mathcal{L}\{y^{(4)}\} - 5^2y(0) - 5^2y'(0) - 5y''(0) - y'''(0) - \mathcal{L}\{y\} = 0$

$\Rightarrow \mathcal{L}\{y^{(4)}\} = \frac{5^2}{5^4 - 1} = \frac{5^2}{8 \times 1 \times (5^4 - 1)} = \frac{5^2 - 6}{5^2 - 1}$

$\Rightarrow y(t) = \frac{1}{2} \sinh t + \frac{1}{2} \sin t$
Forcing functions (the... make a linear ODE nonhomogeneous), are not always continuous. LTs are good for this.

Define $u_c(t) = \begin{cases} 0 & 0 \leq t < c \\ 1 & t \geq c \end{cases}$ as a type of translated Heaviside function. Then $1 - u_c(t)$ is

Then a pulse on $(c, d)$ would be $g(t) = u_c(t) - u_d(t)$

And any modulated signal $f(t)$ defined "turned on" at $t=c \geq 0$ would be written $g(t) = u_c(t)f(t-c)$
Some results for step functions and pulses

1. \( \mathcal{L}\{u_c(t)\} = \int_{0}^{\infty} e^{-st} u_c(t) \, dt = \int_{0}^{\infty} e^{-st} \, dt \)

\[
= \lim_{A \to \infty} \int_{0}^{A} e^{-st} \, dt = \lim_{A \to \infty} \frac{e^{-sA} - 1}{-s} = \frac{e^{-cs}}{s}, \quad s > 0
\]

2. \( \mathcal{L}\{u_c(t) - u_d(t)\} = \frac{e^{-cs}}{s} - \frac{e^{-ds}}{s} = \frac{e^{-cs} - e^{-ds}}{s}, \quad s > 0. \)

3. \( \mathcal{L}\{g(t-c)\} = \mathcal{L}\{u_c(t) g(t-c)\} = e^{-cs} F(s) \)

where \( \mathcal{L}\{f(t)\} = F(s) \), for \( s > a \geq 0. \)

Why?

\[
\mathcal{L}\{u_c(t) g(t-c)\} = \int_{0}^{\infty} e^{-st} u_c(t) g(t-c) \, dt
\]

\[
= \int_{0}^{\infty} e^{-st} f(t-c) \, dt
\]

\[
= \int_{0}^{\infty} e^{-s(t-c)} f(t) \, dt
\]

\[
= e^{-sc} \int_{0}^{\infty} e^{-st} f(t) \, dt = e^{-sc} F(s)
\]

when \( s > a \) as long as \( a > 0. \)

Here is one from \( F(s). \)
**Example**  
Solve the IVP \( g'' - 3g' - 4g = g(t) \)

where \( g(t) \) is the unit pulse function.

\[ g(t) = u_1(t) - u_2(t), \]

and \( y(0) = 1 \), and \( y'(0) = 2 \).

**Solution Note:** One could think of this problem as 3 separate ODEs (IVPs actually), as follows:

1. Solve the IVP \( g'' - 3g' - 4g = 0 \), \( y(0) = 1, y'(0) = 2 \) and then evaluate the endpoint of the solution at \( t = 1 \).
   - Call these \( y_1 \) and \( y_1'(1) = y_1' \).

2. Solve the new IVP \( y'' - 3y' - 4y = 1 \), \( y(1) = y_1, y'(1) = y_1' \). Then evaluate the solution to this ODE at \( t = 2 \). Call it \( y_2 \), \( y_2'(1) = y_2' \).

3. Solve again the ODE \( g'' - 3g' - 4g = 0 \) with new initial conditions \( y(2) = y_2, y'(2) = y_2' \).

4. Stitch together your solutions to a single function.
Laplace Transform solution

Transform the equation to \( \mathcal{L}\{y''-3y'-4y=0\} \)

\[ \mathcal{L}\{y''-3y'-4y\} = \mathcal{L}\{u(t)-u(t-1)\} \]

\[(s^2-3s-4) \mathcal{L}\{y\} + 1 - s = \frac{e^{-s} - e^{-2s}}{s} \]

As in the previous example on the left-hand side.

Solve for \( \mathcal{L}\{y\} \) to set

\[ \mathcal{L}\{y\} = \left( \frac{e^{-s} - e^{-2s}}{s} + s - 1 \right) \left( \frac{1}{s^2-3s-4} \right) \]

\[ = \frac{e^{-s} - e^{-2s}}{s(s-4)(s+1)} + \frac{s-1}{s(s-4)(s+1)} \]

Note that the latter term is just the Laplace transform of the homogeneous solution, and decomposes to

\[ \frac{s-1}{(s-4)(s+1)} = \frac{3}{5} \left( \frac{1}{s-4} \right) + \frac{2}{5} \left( \frac{1}{s+1} \right) \]

Using a partial fraction decomposition on the other term, we set

\[ (e^{-s}e^{-2s}) \left( \frac{1}{s(s-4)(s+1)} \right) = (e^{-s} - e^{-2s}) \left(-\frac{1}{4} \left( \frac{1}{s} \right) + \frac{1}{20} \left( \frac{1}{s-4} \right) + \frac{1}{5} \left( \frac{1}{s+1} \right) \right) \]
Put this together to get

\[ L\{y\} = e^{-s}\left(-\frac{1}{4}\left(\frac{1}{s}\right) + \frac{1}{20}\left(\frac{1}{s^2}\right) + \frac{1}{5}\left(\frac{1}{s^3}\right)\right) \]

\[ - e^{-2s}\left(-\frac{1}{4}\left(\frac{1}{s}\right) + \frac{1}{20}\left(\frac{1}{s^2}\right) + \frac{1}{5}\left(\frac{1}{s^3}\right)\right) \]

\[ + \frac{3}{5}\left(\frac{1}{s^2}\right) + \frac{2}{5}\left(\frac{1}{s^3}\right) \]

Recall that \( L\{e^{st}\} = \frac{1}{s-t} \), and \( L\{u_c(t)\} = \frac{e^{-cs}}{s} \), and \( L\{u_c(t) + (t-c)\} = e^{-cs}F(s) \).

Here on each of the summands above, we set

\[ y(t) = u_1(t)\left(-\frac{1}{4} + \frac{1}{20} e^{4(t-1)} + \frac{1}{5} e^{-(t-1)}\right) \]

\[ - u_2(t)\left(-\frac{1}{4} + \frac{1}{20} e^{4(t-2)} + \frac{1}{5} e^{-(t-2)}\right) \]

\[ + \frac{3}{5} e^{4t} + \frac{2}{5} e^{-t} \]