~~Teach~~ 110.302 Lecture 34:

VIII

Third It helps solve ODEs

<u>example</u> Solve the IVP $y'' - 3y' - 4y = 0$, $y(0) = 1$

$y'(0) = 2$.

<u>Solution</u>: Since LT is linear, we can transform the entire equation:

$$\mathcal{L}\{y'' - 3y' - 4y = 0\}$$

$$\mathcal{L}\{y'' - 3y' - 4y\} = \mathcal{L}\{0\}.$$

$$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} - 4\mathcal{L}\{y\} = 0$$

$$\overbrace{s^2\mathcal{L}\{y\} - s y(0) - y'(0)} - 3\overbrace{\left[s\mathcal{L}\{y\} - y(0)\right]} - 4\mathcal{L}\{y\} = 0$$

$$s^2\mathcal{L}\{y\} - s - 2 - 3s\mathcal{L}\{y\} + 3 - 4\mathcal{L}\{y\} = 0$$

$$\Longrightarrow \quad \mathcal{L}\{y\} = \frac{s-1}{s^2 - 3s - 4} = \frac{s-1}{(s-4)(s+1)} = \frac{a}{s-4} + \frac{b}{s+1}$$

Solve by partial fraction decomp to get

$$\mathcal{L}\{y\} = \frac{3}{5}\left(\frac{1}{s-4}\right) - \frac{2}{5}\left(\frac{1}{s+1}\right)$$

What functions have these as LT?

$$\mathcal{L}\{e^{4t}\} = \frac{1}{s-4}, \quad \mathcal{L}\{e^{-t}\} = \frac{1}{s+1}.$$

$$\Longrightarrow \quad y(t) = \frac{3}{5}e^{4t} + \frac{2}{5}e^{-t}$$

Now solve it the other way in your notes!

Advantages?

1. Works equally well with higher order ODEs.
2. Turns an ODE into an algebraic equation

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3. Many basic functions have well-known LTs (chart page ~~317~~ 321)
4. Incorporates both nonhomogeneous terms and initial data into the calculation,

ex. Solve $\ddot{x} + 3\dot{x} + 2x = 12e^{2t}$ $x(0) = 1$, $\dot{x}(0) = -1$.

Here $\mathcal{L}\{\ddot{x} + 3\dot{x} + 2x = 12e^{2t}\}$.

$$\mathcal{L}\{\ddot{x}\} + 3\mathcal{L}\{\dot{x}\} + 2\mathcal{L}\{x\} = 12\left(\frac{1}{s-2}\right)$$

Finish this calculation:

The solution is $x(t) = e^{2t} + 3e^{-2t} - 3e^{-t}$

ex. Solve $y^{(4)} - y = 0$, $y(0) = y''(0) = y'''(0) = 0$, $y'(0) = 1$.

Here $\mathcal{L}\{y^{(4)} - y = 0\}$ becomes

$$s^4\mathcal{L}\{y\} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - \mathcal{L}\{y\} = 0$$

$$\Rightarrow \mathcal{L}\{y\} = \frac{s^2}{s^4-1} = \frac{s^2}{(s^2-1)(s^2+1)} = \frac{as+b}{s^2-1} + \frac{cs+d}{s^2+1}$$

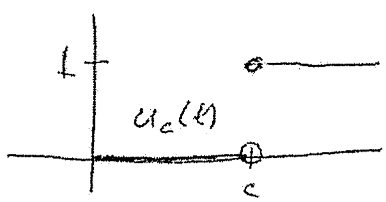
Note: $a = c = 0$, $b = d = \frac{1}{2}$

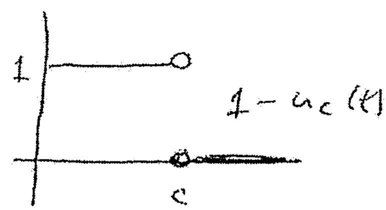
$$= \frac{1}{2}\left(\frac{1}{s^2-1}\right) + \frac{1}{2}\left(\frac{1}{s^2+1}\right)$$

Hence $\boxed{y(t) = \frac{1}{2}\sinh t + \frac{1}{2}\sin t}$

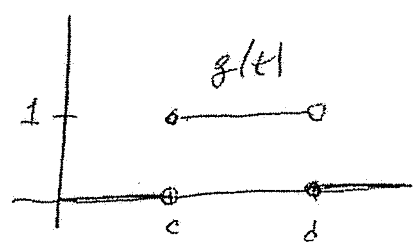
⑤ Forcing functions (the ~~nonhomo~~ $g(t)$ that makes a linear ODE nonhomogeneous), are not always continuous. LTs are good for this.

Define $u_c(t) = \begin{cases} 0 & 0 \le t < c \\ 1 & t \ge c \end{cases}$ a type of translated Heaviside function $(c=0)$. Then $1-u_c(t)$ "

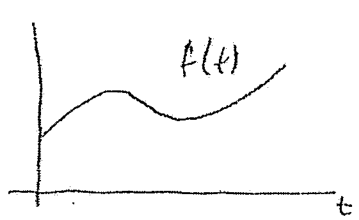


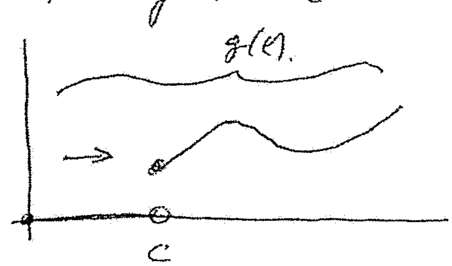


Then a <u>pulse</u> on (c,d) would be $g(t) = u_c(t) - u_d(t)$



And any modulated signal $f(t)$ ~~killed~~ "turned on" @ $t=c>0$ could be written $g(t) = u_c(t) f(t-c)$





Some results for step functions and pulses

(I) $$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\,dt = \int_c^\infty e^{-st}\,dt$$

$$= \lim_{A\to\infty} \int_c^A e^{-st}\,dt = \lim_{A\to\infty} \frac{e^{-sA}}{-s} - \left(\frac{e^{-cs}}{-s}\right)$$

$$= \frac{e^{-cs}}{s}, \quad s>0$$

(II) $$\mathcal{L}\{u_c(t) - u_d(t)\} = \frac{e^{-cs}}{s} - \frac{e^{-ds}}{s} = \frac{e^{-cs} - e^{-ds}}{s}, \quad s>0.$$

(III) $$\mathcal{L}\{g(t)\} = \mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} F(s)$$

where $\mathcal{L}\{f(t)\} = F(s)$, for $s > a \geq 0$.

Why?

$$\mathcal{L}\{u_c(t) f(t-c)\} = \int_0^\infty e^{-st} u_c(t) f(t-c)\,dt$$

$$= \int_c^\infty e^{-st} f(t-c)\,dt$$

$$= \int_0^\infty e^{-s(t+c)} f(t)\,dt$$

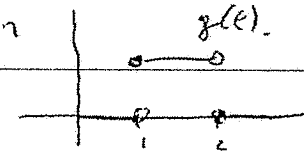
$$= e^{-sc}\int_0^\infty e^{-st} f(t)\,dt = e^{-sc} F(s)$$

when $s>a$ as long as $a \geq 0$.

Here a comes from $F(s)$.

<u>example</u> Solve the IVP $y''-3y'-4y = g(t)$

where $g(t)$ is the unit pulse function



$$g(t) = u_1(t) - u_2(t),$$

and $y(0)=1$, and $y'(0)=2$.

<u>Solution</u> <u>Note</u>: One could think of this problem as 3 separate ODEs (IVPs actually): as follows:

(a) Solve the IVP $y''-3y'-4y=0$, $y(0)=1$, $y'(0)=2$ and then evaluate the endpoint of the solution @ $t=1$. ~~Then~~ Call these $y(1)=y_1$, $y'(1)=y_1'$.

(b) Solve the new IVP $y''-3y'-4y=1$, $y(1)=y_1$, $y'(1)=y_1'$. Then evaluate the solution to this ODE @ $t=2$. call it $y(2)=y_2$, $y'(2)=y_2'$

(c) Solve again the ODE $y''-3y'-4y=0$ with new initial conditions $y(2)=y_2$, $y'(2)=y_2'$.

(d) Stitch together your solutions to a single function.

Laplace Transform solution

Transform the equation to $\mathcal{L}\{y''-3y'-4y\}=u_1(t)-u_2(t)\}$

$$\mathcal{L}\{y''-3y'-4y\}=\mathcal{L}\{u_1(t)-u_2(t)\}$$

$$(s^2-3s-4)\mathcal{L}\{y\}+1-s=\frac{e^{-s}}{s}-\frac{e^{-2s}}{s}=\frac{e^{-s}-e^{-2s}}{s}.$$

as in the previous example on the Left hand side.

Solve for $\mathcal{L}\{y\}$ to get

$$\mathcal{L}\{y\}=\left(\frac{e^{-s}-e^{-2s}}{s}+s-1\right)\left(\frac{1}{s^2-3s-4}\right)$$

$$=\frac{e^{-s}-e^{-2s}}{s(s-4)(s+1)}+\frac{s-1}{(s-4)(s+1)}$$

Note that the latter term is just the Laplace transform of the homogeneous solution, and decomposes to

$$\frac{s-1}{(s-4)(s+1)}=\frac{3}{5}\left(\frac{1}{s-4}\right)+\frac{2}{5}\left(\frac{1}{s+1}\right)$$

Using a partial fraction decomposition on the other term, we get

$$\left(e^{-s}-e^{-2s}\right)\left(\frac{1}{s(s-4)(s+1)}\right)=\left(e^{-s}-e^{-2s}\right)\left(-\frac{1}{4}\left(\frac{1}{s}\right)+\frac{1}{20}\left(\frac{1}{s-4}\right)+\frac{1}{5}\left(\frac{1}{s+1}\right)\right)$$

Put this together to get

$$\mathcal{L}\{y\} = e^{-s}\left(-\frac{1}{4}\left(\frac{1}{s}\right) + \frac{1}{20}\left(\frac{1}{s-4}\right) + \frac{1}{5}\left(\frac{1}{s+1}\right)\right)$$

$$- e^{-2s}\left(-\frac{1}{4}\left(\frac{1}{s}\right) + \frac{1}{20}\left(\frac{1}{s-4}\right) + \frac{1}{5}\left(\frac{1}{s+1}\right)\right)$$

$$+ \frac{3}{5}\left(\frac{1}{s-4}\right) + \frac{2}{5}\left(\frac{1}{s+1}\right)$$

Recall that $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, and $\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}$,

and $\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs}F(s)$.

Hence on each of the summands above, we get

$$y(t) = u_1(t)\left(-\frac{1}{4} + \frac{1}{20} e^{4(t-1)} + \frac{1}{5} e^{-(t-1)}\right)$$

$$- u_2(t)\left(-\frac{1}{4} + \frac{1}{20} e^{4(t-2)} + \frac{1}{5} e^{-(t-2)}\right)$$

$$+ \frac{3}{5} e^{4t} + \frac{2}{5} e^{-t}.$$

Messy, sure! But graph this function (you will need a log plot to really see it). It is a smooth function, even at the pts $t=1$ and $t=2$, and solves $y'' - 3y' - 4y = u_1(t) - u_2(t)$, $y(0)=1$, $y'(0)=2$.