HOMEWORK PROBLEM SET 10: DUE MONDAY, NOVEMBER 12, 2018

110.302 DIFFERENTIAL EQUATIONS
PROFESSOR RICHARD BROWN

Question 1. For the following systems, find a general (real) solution, draw a direction field and plot enough trajectories to fully characterize the nature of the solutions to the system.

(a) $\mathbf{x}' = \begin{bmatrix} -1 & 2 \\ -5 & 1 \end{bmatrix} \mathbf{x}$.

(b) $\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \mathbf{x}$.

Question 2. Solve the IVP

$$\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

and, in detail, describe the behavior of the solution, both near $t = 0$ as well as when t goes to infinity and minus infinity. For instance, what is the value of t when it is closest to the origin? And does the solution have any asymptotes? If it has asymptotes, describe the nature of these asymptotes.

Question 3. Do the following for the ODE system $\mathbf{x}' = \begin{bmatrix} \alpha & -1 \\ 1 & \alpha \end{bmatrix} \mathbf{x}$:

(a) Determine the eigenvalues as functions of α.

(b) Find the critical values of α, defined as values of α where the qualitative nature of the phase portrait for the system changes.

(c) Draw the phase portrait for values of α slightly larger and slightly smaller than each critical value of α.

Question 4. A mass m on a spring with spring constant k satisfies the differential equation

$$mu'' + ku = 0,$$

where $u(t)$ is the displacement at time t of the mass from its equilibrium position.

(a) Let $x_1 = u$ and $x_2 = u'$ and show that the resulting first-order system is

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \mathbf{x}.$$

(b) Solve the system and draw a phase portrait.

(c) For one non-trivial trajectory (not the origin), draw the corresponding component functions as functions of time.

(d) Determine the relationship between the eigenvalues of the coefficient matrix and the frequency of the spring-mass system.