HOMEWORK PROBLEM SET 12: DUE NOVEMBER 30, 2018

110.302 DIFFERENTIAL EQUATIONS
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Note: If you are asked to draw for any of these problems, feel free to use any technology you can find. One way is the ODE visualization software from Bluffton University here:

`https://bluffton.edu/homepages/facstaff/nesterd/java/slopefields.html`,

or the JOde software here:

`http://www.math.jhu.edu/mathcourses/302/jode/JOdeApplet2DFramed.html`.

In the latter case, you will have to (1) find a way for your browser to run Java (the extension IETab works in Chrome), (2) load Java on your computer, (3) bypass security by placing the website http://www.math.jhu.edu into a white list in the Java Settings.

Question 1. For each of the following systems, do the following: (1) calculate the eigenvalues and eigenvectors; (2) classify the equilibrium at the origin, both in its type and its stability; and (3) Sketch a phase portrait.

(a) $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \mathbf{x}$.

(b) $\mathbf{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \mathbf{x}$.

(c) $\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x}$.

(d) $\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \mathbf{x}$.

Question 2. The linear, second-order, homogeneous, autonomous, ODE with constant coefficients

$$m\frac{d^2u}{dt^2} + c\frac{du}{dt} + ku = 0,$$

where $m, c, k > 0$ are all constants, models a spring-mass system with damping. Write this ODE as a first-order system, where $x = u$, and $y = u'$. Show that the origin ($x = 0$ and $y = 0$) is a critical point, and analyze the nature and stability of this critical point as a function of the parameters m, c, and k.

Question 3. (What Figure 9.1.9 on page 507 of the text means.) Given the ODE system

$$\mathbf{x}' = A\mathbf{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x},$$

let $p = \operatorname{tr} A = a + d$, and let $q = \det A = ad - bc$, and finally let $\Delta = p^2 - 4q$. Show that the critical point at the origin is a:

(a) Node if $q > 0$ and $\Delta \geq 0$;

(b) Saddle if $q < 0$;

(c) Spiral Node if $p \neq 0$ and $\Delta < 0$;

(d) Center if $p = 0$ and $q > 0$.

(e) Also show that, if the two solutions to the characteristic equation of A are r_1 and r_2, that it is always the case that $p = r_1 + r_2$ and $q = r_1 r_2$.

(f) Show that the critical point at the origin is asymptotically stable if $q > 0$ and $p < 0$.

(g) Show that the critical point at the origin is stable if $q > 0$ and $p = 0$.

(h) Show that the critical point at the origin is unstable if $q < 0$ or $p > 0$.

Question 4. For each system, (1) find all critical points, (2) use a computer or hand draw a direction field and sketch a few trajectories near the critical points, and (3) determine as best as one can the type and stability of each critical point.

(a) $\frac{dx}{dt} = -xy + x, \quad \frac{dy}{dt} = y + 2xy.$

(b) $\frac{dx}{dt} = 2x - x^2 - xy, \quad \frac{dy}{dt} = 3y - 2y^2 - 3xy.$

Question 5. For each system, (1) find a function $H(x, y) = c$ satisfied by the trajectories, and (2) plot several level curves of H, indicting the direction of travel for increasing t.

(a) $\frac{dx}{dt} = 2y, \quad \frac{dy}{dt} = 8x.$

(b) $\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\sin x$ (this is the undamped pendulum.)

Question 6. For the following systems, verify that $\mathbf{x}^o = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an equilibrium and that the system is locally linear at the origin. Then classify the type and stability of the origin as best as one can by locally linearizing the system at the origin and classifying the linear equilibrium.

(a) $\dot{x} = x - y^2, \quad \dot{y} = x - 2y + x^2.$

(b) $\dot{x} = 2xy + y - x, \quad \dot{y} = x^2 - (y^2 + y + 4x).$

Question 7. For the system $\dot{x} = \epsilon x + y$ and $\dot{y} = -x + \epsilon y$, where ϵ is a parameter, classify the type and stability of the origin for different values of ϵ, and use this to verify that a center is not structurally stable; that neither the type nor the stability of a center persists under perturbations.

Question 8. For the competing species model $\dot{x} = x\left(\frac{3}{2} - \frac{1}{2}x - y\right)$ and $\dot{y} = y\left(2 - y - \frac{9}{8}x\right)$, draw a phase portrait and discuss the limiting behavior of the species populations $x(t)$ and $y(t)$ as $t \to \infty$ for various initial population sizes. Now do the same for the predator-prey model $\dot{x} = x\left(1 - \frac{y}{2}\right)$ and $\dot{y} = y\left(\frac{x}{2} - \frac{1}{4}\right)$.

Question 9. For the system $\dot{x} = y$ and $\dot{y} = x + 2x^3$, do the following:

(a) Show that the origin is a saddle.

(b) Sketch a phase portrait for the linearized system and show that all of the trajectories of the linear system that tend to the origin are on the line $y = -x$.

(c) Sketch some trajectories of the nonlinear system integrating the corresponding first-order equation in $\frac{dy}{dx}$. In particular, sketch the trajectories that are asymptotic to the lines $y = x$ and $y = -x$.