**HOMEWORK PROBLEM SET 13: NOT TO BE HANDED IN.**

110.302 DIFFERENTIAL EQUATIONS
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**Question 1.** For each ODE system, find and classify all equilibria and cycles and draw a phase portrait:

**(a)** $\dot{r} = r(1-r)(r-2)(3-r)(1+r), \quad \dot{\theta} = -1.$

**(b)** $\dot{r} = r(1-r)^2(r-2), \quad \dot{\theta} = 4.$

**Question 2.** Transform the system

$$\dot{x} = x - y - x(x^2+y^2)$$
$$\dot{y} = x + y - y(x^2+y^2)$$

into a system in polar coordinates to get $\dot{r} = r(1-r^2)$ and $\dot{\theta} = 1$.

**Question 3.** Now for the system

$$\dot{x} = \alpha x - y - x(x^2+y^2)$$
$$\dot{y} = x + \alpha y - y(x^2+y^2),$$

where $\alpha$ is a parameter, do the following:

**(a)** Show the origin is the only critical point for all values of $\alpha \in \mathbb{R}$.

**(b)** Linearize the system at the origin and use it to determine the type and stability of the nonlinear equilibrium. How does this classification depend on $\alpha$?

**(c)** Transform the system into polar coordinates, and explain how the phase portrait changes as the values of $\alpha$ change. Locate any bifurcation values of $\alpha$ and describe and draw a representative phase portrait on either side of each bifurcation value. (Note that the bifurcation you see here is called a *Poincaré-Andropov-Hopf bifurcation* or simply a *Hopf bifurcation.*)

**Question 4.** Determine the periodic solutions, if any, of the system

$$\dot{x} = y + \frac{x}{\sqrt{x^2+y^2}}(x^2+y^2-2), \quad \dot{y} = -x + \frac{y}{\sqrt{x^2+y^2}}(x^2+y^2-2).$$

**Question 5.** For the following, find the function that transforms to the expression given:

**(a)** $F(s) = \frac{3}{s^2+4}$.

**(b)** $F(s) = \frac{2}{s^2+3s-4}$.

**(c)** $F(s) = \frac{4}{(s-1)^3}$.

**Question 6.** For the following, Use the Laplace Transform to solve the following IVPs:

**(a)** $y'' - y' - 6y = 0, \quad y(0) = 1, \quad y'(0) = -1.$

**(b)** $y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.$

**(c)** $y^{(4)} - y = 0, \quad y(0) = y''(0) = 1, \quad y'(0) = y'''(0) = 0.$

**Question 7.** Find the Laplace transform of the function $y(t)$ that satisfies the IVP

$$y'' + y = \begin{cases} t & 0 \leq t < 1 \\ 2-t & 1 \leq t < 2 \\ 0 & t \geq 2, \end{cases} \qquad y(0) = y'(0) = 0.$$

You do not need to find $y(t)$.

**Question 8.** Sketch $g(t)$ on the interval $t \geq 0$ and find its Laplace Transform:

**(a)** $g(t) = u_1(t) + 2u_2(t) - 6u_4(t)$.

**(b)** $g(t) = f(t-2)u_2(t)$, where $f(t) = t^2$.

**(c)** $g(t) = u_2(t)(t-3) - (t-2)u_3(t)$.

**Question 9.** Solve the IVPs: [This is optional. But give these a try!]

**(a)** $y'' + 3y' + 2y = h(t), \quad y(0) = y'(0) = 0, \quad h(t) = \begin{cases} 1 & 0 \leq t < 10 \\ 0 & t \geq 10. \end{cases}$

**(b)** $y'' + 2y' + 2y = f(t), \quad y(0) = 0, \quad y'(0) = 1, \quad f(t) = \begin{cases} 1 & \pi \leq t < 2\pi \\ 0 & 0 \leq t < 10 \text{ and } t \geq 2\pi. \end{cases}$

**Question 10.** For the given first-order IVPs, do the following: (1) Approximate the solution at $t = 2$ by using Euler's Method with a step size of $h = .5$, (2) solve the ODE and calculate the difference between your approximate solution and the actual solution.

**(a)** $y' = 3 + t - y, \quad y(0) = 1.$

**(b)** $y' = 2y - 3t, \quad y(0) = 1.$

**Question 11.** Suppose that $x(t)$ solves the ODE $\dot{x} = \sqrt{x+t}$. Use Euler's Method to approximate $x(4)$ knowing $x(1) = 3$. Use a step size of $h = .5$.