Question 1. For each ODE system, find and classify all equilibria and cycles and draw a phase portrait:

(a) \( \dot{r} = r(1 - r)(r - 2)(3 - r)(1 + r), \quad \dot{\theta} = -1. \)

(b) \( \dot{r} = r(1 - r)^2(r - 2), \quad \dot{\theta} = 4. \)

Question 2. Transform the system

\[
\begin{align*}
\dot{x} &= x - y - x(x^2 + y^2) \\
\dot{y} &= x + y - y(x^2 + y^2)
\end{align*}
\]

into a system in polar coordinates to get \( \dot{r} = r(1 - r^2) \) and \( \dot{\theta} = 1. \)

Question 3. Now for the system

\[
\begin{align*}
\dot{x} &= \alpha x - y - x(x^2 + y^2) \\
\dot{y} &= x + \alpha y - y(x^2 + y^2),
\end{align*}
\]

where \( \alpha \) is a parameter, do the following:

(a) Show the origin is the only critical point for all values of \( \alpha \in \mathbb{R} \).

(b) Linearize the system at the origin and use it to determine the type and stability of the nonlinear equilibrium. How does this classification depend on \( \alpha \)?

(c) Transform the system into polar coordinates, and explain how the phase portrait changes as the values of \( \alpha \) change. Locate any bifurcation values of \( \alpha \) and describe and draw a representative phase portrait on either side of each bifurcation value. (Note that the bifurcation you see here is called a Poincaré-Andropov-Hopf bifurcation or simply a Hopf bifurcation.)

Question 4. Determine the periodic solutions, if any, of the system

\[
\begin{align*}
\dot{x} &= y + \frac{x}{\sqrt{x^2 + y^2}}(x^2 + y^2 - 2), \\
\dot{y} &= -x + \frac{y}{\sqrt{x^2 + y^2}}(x^2 + y^2 - 2).
\end{align*}
\]

Question 5. For the following, find the function that transforms to the expression given:

(a) \( F(s) = \frac{3}{s^2+4}. \)

(b) \( F(s) = \frac{2}{s^2+3s-4}. \)

(c) \( F(s) = \frac{4}{(s-1)^4}. \)

Question 6. For the following, Use the Laplace Transform to solve the following IVPs:

(a) \( y'' - y' - 6y = 0, \quad y(0) = 1, \quad y'(0) = -1. \)
(b) \( y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1. \)

(c) \( y^{(4)} - y = 0, \quad y(0) = y''(0) = 1, \quad y'(0) = y'''(0) = 0. \)

**Question 7.** Find the Laplace transform of the function \( y(t) \) that satisfies the IVP

\[
y'' + y = \begin{cases} 
  t & 0 \leq t < 1 \\
 2 - t & 1 \leq t < 2 \\
 0 & t \geq 2,
\end{cases} \quad y(0) = y'(0) = 0.
\]

You do not need to find \( y(t) \).

**Question 8.** Sketch \( g(t) \) on the interval \( t \geq 0 \) and find its Laplace Transform:

(a) \( g(t) = u_1(t) + 2u_2(t) - 6u_4(t) \).

(b) \( g(t) = f(t - 2)u_2(t) \), where \( f(t) = t^2 \).

(c) \( g(t) = u_2(t)(t - 3) - (t - 2)u_3(t) \).

**Question 9.** Solve the IVPs: [This is optional. But give these a try!]

(a) \( y'' + 3y' + 2y = h(t), \quad y(0) = y'(0) = 0, \quad h(t) = \begin{cases} 
  1 & 0 \leq t < 10 \\
  0 & t \geq 10.
\end{cases} \)

(b) \( y'' + 2y' + 2y = f(t), \quad y(0) = 0, \quad y'(0) = 1, \quad f(t) = \begin{cases} 
  1 & \pi \leq t < 2\pi \\
  0 & 0 \leq t < 10 \text{ and } t \geq 2\pi.
\end{cases} \)

**Question 10.** For the given first-order IVPs, do the following: (1) Approximate the solution at \( t = 2 \) by using Euler’s Method with a step size of \( h = .5 \), (2) solve the ODE and calculate the difference between your approximate solution and the actual solution.

(a) \( y' = 3 + t - y, \quad y(0) = 1. \)

(b) \( y' = 2y - 3t, \quad y(0) = 1. \)

**Question 11.** Suppose that \( x(t) \) solves the ODE \( \dot{x} = \sqrt{x + t} \). Use Euler’s Method to approximate \( x(4) \) knowing \( x(1) = 3 \). Use a step size of \( h = .5 \).