HOMEWORK PROBLEM SET 3: DUE SEPTEMBER 21, 2018

110.302 DIFFERENTIAL EQUATIONS
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Question 1. Without solving the IVPs, determine the largest interval in which the solution is guaranteed to exist for each initial value.

(a) $x(x-4)y' + y = 0$, for **(i)** $y(2) = 1$, **(ii)** $y(-2) = 1$, and **(iii)** $y(0) = 1$.

(b) $(\ln t)\dot{z} + z - \tan t = 0$, for **(i)** $z(2) = 0$, **(ii)** $z(1) = 3$, and **(iii)** $z(-\pi) = 1$.

Question 2. State where in the ty-plane the solutions to the following ODEs are guaranteed to exist. Then also state where solutions are also guaranteed to be uniquely defined.

(a) $y'(1 - t^2 + y^2) = \ln |ty|$.

(b) $y' = \dfrac{y - t}{5y + 2t}$.

Question 3. Solve the IVP $z' = -z^3$, for $z(0) = z_0$ and determine how the interval in which the solutions exists depends on the initial value z_0.

Question 4. (Linear vs. non-linear) Show that $\varphi(t) = e^{-4t}$ is a solution to the linear ODE $y' + 4y = 0$, and also that $y = c\varphi(t)$ is also a solution for any $c \in \mathbb{R}$ a constant. Then show that $\varphi(t) = \frac{1}{t}$ is a solution to the non-linear ODE $y' + y^2 = 0$ on the interval $t > 0$, but that $y = c\varphi(t)$ is not a solution for any c other than $c = 0$ or $c = 1$. This is one of the special properties of certain linear ODEs and is part of what is called the *Principle of Superposition*. This will be introduced in Chapter 3).

Question 5. For the following autonomous ODEs in the format $y' = f(y)$, do the following: (1) sketch the graph of $f(y)$ verses y, (2) determine the critical points (the places where equilibrium solutions exist) and classify each equilibrium as asymptotically stable, semi-stable, or unstable, (4) draw a phase line, and (5) draw enough trajectories in the ty-plane to completely exhibit solution behavior.

(a) $y' = y(y+1)(y-2)$.

(b) $y' = a\sqrt{y} - by$, where $a > 0$, $b > 0$, and $y \geq 0$.

(c) $y' = y^2(1 - y^2)$.

(d) $y' = y^2(4 - y)^2$.

Question 6. Suppose y_1 is a critical point of the ODE $\frac{dy}{dt} = f(y)$. Show that the equilibrium solution $y(t) \equiv y_1$ is asymptotically stable, a sink, if $\frac{df}{dy}\big|_{y=y_1} < 0$ and unstable, a source, if $\frac{df}{dy}\big|_{y=y_1} > 0$.