HOMEWORK PROBLEM SET 6: DUE OCTOBER 12, 2018

110.302 DIFFERENTIAL EQUATIONS
PROFESSOR RICHARD BROWN

Question 1. Use Euler's formula to do the following:

- **(a)** Write $e^{i\pi}$ in the form $a + ib$.
- **(b)** Write 2^{2-2i} in the form $a + ib$.
- **(c)** Write π^{2i-1} in the form $a + ib$.
- **(d)** Show that

$$\cos t = \frac{e^{it} + e^{-it}}{2}, \quad \text{and} \quad \sin t = \frac{e^{it} - e^{-it}}{2i}.$$

Question 2. Solve the following:

- **(a)** $y'' - 6y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 2.$
- **(b)** $9y'' + 9y' - 4y = 0.$
- **(c)** $y'' + 2y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = 4.$
- **(d)** $2y'' + 2y' + y = 0.$
- **(e)** $2y'' + 2y' = 0.$
- **(f)** $y'' = -a\left(ay + 2y'\right)$, $a \in \mathbb{R}$.

Question 3. For the ODE $ay'' + by' + cy = 0$, suppose $b^2 - 4ac < 0$, so that the two solutions to the characteristic equation for the ODE are $\lambda \pm i\mu$. Do the following:

- **(a)** Show that $u(t) = e^{\lambda t} \cos \mu t$ and $v(t) = e^{\lambda t} \sin \mu t$ are, in fact, solutions to the ODE.
- **(b)** Show that the Wronskian $W(u(t), v(t)) = \mu e^{2\lambda t}$, thereby establishing the linear independence of $u(t)$ and $v(t)$ on all of $\mathbb{R}$.
- **(c)** Do the same for the ODE in the case that $b^2 = 4ac$ by showing that $u(t) = e^{-\frac{b}{2a}t}$ and $v(t) = te^{-\frac{b}{2a}t}$ each solve the ODE and are linearly independent of each other as functions.

Question 4. For the IVP $y'' = \frac{-4y - 12y'}{9}, \quad y(0) = \alpha > 0, \quad y'(0) = -1$, do the following:

- **(a)** Find all value(s) of α where $\lim_{t \to \infty} y(t) = 0$.
- **(b)** Find all value(s) of α where $y(t) > 0$, for all $t \in \mathbb{R}$.

Question 5. Given the following ODEs and the one solution given, write out a full fundamental set of solutions:

- **(a)** $t^2 y'' + 2ty' = 2y, \quad t > 0, \quad y_1(t) = t.$
- **(b)** $(x - 1)y'' - xy' + y = 0, \quad x > 1, \quad y_1(x) = e^x.$

Question 6. Solve the following using the Undetermined Coefficients Method:

(a) $y'' + 4y = t^2 + 3e^t$.

(b) $\ddot{x} + 2\dot{x} + 5x = 4e^{-t}\cos 2t, \quad x(0) = 1, \quad \dot{x}(0) = 0$.

(c) $y'' - y' - 2y = \cosh 2x$. Hint: Write the hyperbolic cosine function as a linear combination of exponentials.