Differential Equations – Singular Solutions

Consider the first-order separable differential equation: \( \frac{dy}{dx} = f(y)g(x) \). \( (1) \)

We solve this by calculating the integrals: \( \int \frac{dy}{f(y)} = \int g(x)dx + C. \) \( (2) \)

If \( y_0 \) is a value for which \( f(y_0) = 0 \), then \( y = y_0 \) will be a solution of the above differential equation \( (1) \). We call the value \( y_0 \) a critical point of the differential equation and \( y = y_0 \) (as a constant function of \( x \)) is called an equilibrium solution of the differential equation.

If there is no value of \( C \) in the solution formula \( (2) \) which yields the solution \( y = y_0 \), then the solution \( y = y_0 \) is called a singular solution of the differential equation \( (1) \).

The “general solution” of \( (1) \) consists of the solution formula \( (2) \) together with all singular solutions.

Note: by “general solution”, I mean a set of formulae that produces every possible solution.

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**Example 1:** Solve: \( \frac{dy}{dx} = (y - 3)^2. \) \( (3) \)

**Solution:** \( \int \frac{dy}{(y - 3)^2} = \int dx. \) Thus, \( \frac{-1}{y - 3} = x + C; \) \( y - 3 = \frac{-1}{x + C}; \) and \( y = 3 - \frac{1}{x + C}, \) \( (4) \)

where \( C \) is an arbitrary constant.

Both sides of the DE \( (3) \) are zero when \( y = 3 \). No value of \( C \) in \( (4) \) gives \( y = 3 \) and thus, the solution \( y = 3 \) is a singular solution.

The general solution of \( (3) \) consists of: \( y = 3 - \frac{1}{x + C} \) (\( C \) is an arbitrary constant) and \( y = 3. \)

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See over \( \Rightarrow \)
Example 2: Solve: \[ \frac{dy}{dx} = y^2 - 4. \] (5) 

Solution: \[ \int \frac{dy}{y^2 - 4} = \int dx. \] Using partial fractions,

\[ \int \frac{dy}{(y - 2)(y + 2)} = \int \left[ \frac{1}{4} \left( \frac{1}{y - 2} + \frac{-1}{y + 2} \right) \right] dy = \int dx. \]

Thus, \[ \int \left[ \frac{1}{y - 2} + \frac{-1}{y + 2} \right] dy = \int 4 dx. \]

Integrating, \[ \ln(y - 2) - \ln(y + 2) = 4x + C. \]

Taking exponentials, \[ \frac{y - 2}{y + 2} = e^{4x} C_1 = s \text{ (say).} \]

Then,
\[
\begin{align*}
y - 2 &= s(y + 2) = sy + 2s \\
y - sy &= 2 + 2s \\
y(1 - s) &= 2 + 2s \\
y &= \frac{2 + 2s}{1 - s}.
\end{align*}
\]

Thus,
\[ y = \frac{2 + 2C_1 e^{4x}}{1 - C_1 e^{4x}}, \] (6)

where \( C_1 \) is an arbitrary constant.

Both sides of the DE (5) are zero when \( y = \pm 2 \). If we put \( C_1 = 0 \) in (6), we obtain the solution: \( y = 2 \). However, no value of \( C_1 \) in (6) gives \( y = -2 \) and thus, the solution \( y = -2 \) is a singular solution.

The general solution of (5) consists of:
\[ y = \frac{2 + 2C_1 e^{4x}}{1 - C_1 e^{4x}} \text{ (} C_1 \text{ is an arbitrary constant) and } y = -2. \]