Section 6.2

Green's Theorem Let $\mathcal{D}$ be a closed, bounded region in $\mathbb{R}^2$, whose boundary $\partial\mathcal{D}$ is a finite union of simple, closed curves, oriented so that $\mathcal{D}$ is always on the left.

For a $C^1$-vector field on $\mathcal{D}$

$$\mathbf{F}(x,y) = M(x,y) \mathbf{i} + N(x,y) \mathbf{j},$$

we have

$$\oint_{\partial\mathcal{D}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{D}} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy$$

or

$$\iint_{\mathcal{D}} \left( \nabla \times \mathbf{F} \right) \cdot \mathbf{k} \, dA.$$

Note 1. a) is obvious since $\mathbf{F} = [M] \mathbf{i} + [N] \mathbf{j}, \quad d\mathbf{s} = [dx] \mathbf{i} + [dy] \mathbf{j}$

b) is also obvious since $(\nabla \times \mathbf{F}) \cdot \mathbf{k} = \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$

To see this, think of a vector field in $\mathbb{R}^2$ as a vector field in $\mathbb{R}^3$ with 2-component 0. Then calculate $\nabla \times \mathbf{F}$. 
Notes (cont'd)

2. The theorem basically says that the vector line integral (the circulation) of $\vec{F}$ along $\partial D$ equals the curl of $\vec{F}$ on $D$.

\[ \oint_{\partial D} \vec{F} \cdot d\vec{s} \] measures the aggregate component of $\vec{F}$ tangent to $\vec{e}$

- curl in 2-d measures in $\mathbb{R}^2$ the net total one would feel if flowing along $\vec{F}$.

So, the sum total of the push or pull of a particle by $\vec{F}$ along $\partial D$ equals the total net total effect of $\vec{F}$ on $D$.

ex. For a constant vector field $\vec{F}$

- $\nabla \times \vec{F} = \vec{0}$, so RHTS = 0.
- What happens on LHTS?
- What happens in middle?
Notes (cont'd)

3. The proof is elementary, and relies on 3 facts:
   
   a. **Lemma 1.** \( D \) is elementary of type I, then \( \oint_D M \, dx = \int_D \frac{\partial N}{\partial y} \, dA \)
   
   b. **Lemma 2.** \( D \) is elementary of type II, then \( \oint_D N \, dy = \int_D \frac{\partial M}{\partial x} \, dA \)

   c. Any region \( D \), valid for Green's Theorem, can be cut up into a finite # of elementary regions, so that
      
      i. The ends of each cut line in \( D \)
      ii. The cuts do not intersect
      iii. \( D = \bigcup D_i \) with each \( D_i \) elem. of same type.
      iv. Each cut intersects exactly 2 \( D_i \)'s with each cut oriented in each \( D_i \) oppositely.

   **Note:** Each vector line integral along cuts will cancel out. So no contribution of cuts in the calculation. And for the double integral, the cuts provide no contribution either.
Idea of proof of Lemma 1

Lemma 16 D is elementary of type I.

\[ \int \mathbb{M} \, dx = - \int_{D} \frac{\partial M}{\partial y} \, dA \]

\[ \text{Proof: } \quad D = \{(x,y) \in \mathbb{R}^2 \mid x \leq b \land 0 \leq y \leq \beta(x) \} \]

\[ \text{Orient } dD = c_1^+ u_1 \, dx_1^+ u_2 \, dx_2^- \]

as needed. (plus means c path with variables, minu c\text{c} contra c.)

Recall (RHs)

\[ -\int_{D} \frac{\partial M}{\partial y} \, dA = \int_{c} \int_{\alpha(x)}^{b} \alpha(x) \, dy \, dx \]

\[ = \int_{c}^{b} \left( M(x, \beta(x)) - M(x, \alpha(x)) \right) \, dx \]

by FTC

\[ = \int_{c}^{b} \left( M(x, \alpha(x)) - M(x, \beta(x)) \right) \, dx \]

Here

\[ -\int_{c}^{b} M(x, \alpha(x)) \, dx = \int_{c}^{b} M \, dx \]

\[ \int_{c}^{b} M(x, \alpha(x)) \, dx = \int_{c}^{b} M \, dx \]
Def. A C-regular field $F$ has

2 pre-sunk C-curves $y = ax^2 + bx + c$, $x = dy^2 + ey + f$.

We start with a different

Section 6.3 Curves: The field

Exercise: Prove Lemma 2.

Thus

$\int_{a}^{b} \frac{dx}{x^2 + \sqrt{x^2 + 1}} = \left. \frac{1}{\sqrt{1-x^2}} \right|_{a}^{b}$

And

$\int_{a}^{b} \frac{1}{x^2 + \sqrt{x^2 + 1}}$
Notice that since \( \vec{x}_1, \vec{x}_2 \) have some ends, they together form a closed curve (considering \( \vec{x}_2 \) backwards, Part ii).

Sometimes this closed curve is simple.

**Theorem.** A \( C^0 \)-vector field \( \vec{F} \) has path-independent line integrals if \( \int_C \vec{F} \cdot d\vec{s} = 0 \) for piecewise \( C^1 \) simple closed curves \( C \) in domain of \( \vec{F} \).

**Note:** \( \int_C \vec{F} \cdot d\vec{s} \) is not simple and consists of a finite # of isolated intervals that coincide. Still true! (How?)
A $C^0$-vector field $\mathbf{F}$ is called **conservative** or a **gradient field** of a $C^1$-real-valued function $f$ if $\nabla f = \mathbf{F}$. Such an $f$ is called a potential for $\mathbf{F}$.

**Notes**

1. Conservative vector fields always have path-independent line integrals.

\begin{align*}
\int_{\mathbf{x}}^{\mathbf{x}'} \mathbf{F} \cdot d\mathbf{s} &= \int_{\mathbf{x}}^{\mathbf{x}'} \nabla f \cdot d\mathbf{s} \\
&= \int_{t=a}^{t=b} \frac{d}{dt} \left[ f(\mathbf{x}(t)) \right] dt \\
&= f(\mathbf{x}(b)) - f(\mathbf{x}(a))
\end{align*}

This is the §6.3.3.
In $\mathbb{R}^2$ or $\mathbb{R}^3$, a conservative vector field is irrotational. If $\mathbf{F}$ is conservative, then
\[ \nabla \times \mathbf{F} = \nabla \times \nabla f = 0. \]

Converse: If $\mathbf{F}$ is irrotational, and domain is simply connected, then $\mathbf{F}$ is conservative.

This is Thm 6.3.5.

Def. A region is simply connected if it is connected (i.e., in one piece) and every simple closed curve in region has entire interior in region $R$. - simply connected

$R_2$ - not simply connected
Ex. Let \( \vec{F} = \frac{-y}{x^2+y^2} \hat{i} + \frac{x}{x^2+y^2} \hat{j} + 0 \hat{k} \) on \( \mathbb{R}^3 - \{(0,0,2) \in \mathbb{R}^3 \mid z \in \mathbb{R} \} \). Has no curl \( \nabla \times \vec{F} = \vec{0} \) but \( \vec{F} \) is not conservative. Here \( W \) is not simply conn.

Q: How to tell if a \( \vec{F} \) is conservative otherwise?

A: Check mixed partials of gradient

\[ \begin{align*}
\text{1.} & \quad \text{Check mixed partials of } \nabla \Phi \\
\text{2.} & \quad \text{Intersect } \phi \text{ to find } \Phi
\end{align*} \]

Ex. (6.3.14) Find a potential for

\[ \vec{F} = \begin{bmatrix} y + z \frac{1}{x^2+y^2} \\ z \frac{1}{x^2+y^2} \\ 0 \end{bmatrix} \]

is conservative.

Soln: \( \nabla \Phi \end{bmatrix} = y + z \), so \( \Phi(x,y,z) = xy + x^2 + g(y,z) \)

\[ \begin{align*}
\frac{\partial}{\partial \hat{x}} = x + \frac{z}{x^2+y^2} (y+2) = 2z \\
\nabla \times \vec{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
y + 2z & x + 2 & 0
\end{vmatrix} = (1-2)^2 \hat{k} \text{ not conservative} \\
\n\nabla \times \vec{F} \neq \vec{0}
\end{align*} \]
ex. Find a potential for

\[ \vec{F} = (2x+y) \hat{i} + (2 \cos yz + x) \hat{j} + (y \cos yz) \hat{k} \]

**Sol.** Here \( \frac{\partial F}{\partial x} = 2x + y, \) so \[ H(x,y,z) = \int \frac{\partial F}{\partial x} \, dx = x^2 + xy + g(y,z). \]

Here \( \frac{\partial F}{\partial y} = x + 2 \cos yz \)

\[ \Rightarrow \frac{\partial g}{\partial y} (y,z) = 2 \cos yz \]

\[ \Rightarrow g(y,z) = \sin yz + h(z) \]

\[ \Rightarrow H(x,y,z) = x^2 + xy + \sin yz + h(z) \]

\[ \frac{\partial F}{\partial z} = y \cos yz + h'(z) = y \cos yz \]

\[ \Rightarrow h'(z) = 0 \Rightarrow h(z) = \text{const}. \]

\[ \Rightarrow H(x,y,z) = x^2 + xy + \sin(yz) \]