Section 7.2

In the same way that we integrated functions (real-valued) and vector fields over curves, we can do so on surfaces.

I Real-valued (scalar) functions

- Like for scalar line integrals, if the surface \( \mathbf{S} \) in \( \mathbb{R}^3 \) is inside the domain of a real-valued function in \( \mathbb{R}^3 \), we can restrict the domain to the surface and integrate.
- If we parameterize the surface with coordinates on the surface, this is like a double integral.
- However, the resulting value should be parameter independent.
- Basically, we look to define
  \[ \iint_{\mathbf{S}} f \, d\mathbf{S}, \text{ where } d\mathbf{S} = \lVert N(s,t) \rVert \, ds \, dt \]
  is a surface differential.
Def. Let \( \mathbf{X} : \mathcal{D} \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) be a smooth parametric surface, where \( \mathcal{D} \) is bounded. Let \( f \) be a \( C^0 \) function on a domain that includes \( \mathbf{X} \). Then the scalar surface integral of \( f \) along \( \mathbf{X} \) is

\[
\iint_{\mathcal{D}} f(\mathbf{X}(s,t)) \left\| \mathbf{X}_s \times \mathbf{X}_t \right\| \, ds \, dt
\]

\[
= \iint_{\mathcal{D}} f(x(s,t), y(s,t), z(s,t)) \sqrt{\left( \frac{dx}{ds} \right)^2 + \left( \frac{dy}{ds} \right)^2 + \left( \frac{dz}{ds} \right)^2} \, ds \, dt
\]

Notes:
1. Like for line integrals, \( ds \) is a scalar 2-form (\( ds \) as a scalar 1-form) and represents a infinitesimal element in surface area along the surface.
2. For \( f(x,y,z) = 1 \), this integral gives the surface area of \( \mathbf{X}(\mathcal{D}) \).
3. In coordinates \( (s,t) \), this looks like a standard double integral.
4. If \( \mathbf{X} \) is not smooth but has edges (piecewise smooth) then each smooth piece must be integrated separately and the results added together.
Def. Let $\mathbf{X} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth normal surface, where $D$ is bounded. Let $\mathbf{F}$ be a $C^1$-vector field on a domain in $\mathbb{R}^3$ that includes $\mathbf{X}(D)$. Then the vector surface integral

$\mathbf{F}$ along $\mathbf{X}$ is

$$
\iint_{\mathcal{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_{D} \mathbf{F}(\mathbf{X}(s,t)) \cdot \mathbf{N}(s,t) \, ds \, dt
$$

$$
= \iint_{D} \mathbf{F}(x(s,t), y(s,t), z(s,t)) \cdot \begin{bmatrix}
\frac{\partial y}{\partial t} & \frac{\partial y}{\partial s} & 0 \\
0 & \frac{\partial z}{\partial t} & \frac{\partial z}{\partial s} \\
1 & 0 & 0
\end{bmatrix}
\, ds \, dt.
$$

Notes:

1. Here $d\mathbf{S} = \mathbf{N}(s,t) \, ds \, dt$ is a vector 2-form.

2. If we normalize the normal vector

$$
\mathbf{N}(s,t) = \frac{\mathbf{N}(s,t)}{\|\mathbf{N}(s,t)\|}, \quad \text{then}
$$

$$
\iint_{\mathcal{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_{D} \mathbf{F}(\mathbf{X}(s,t)) \cdot \frac{\mathbf{N}(s,t)}{\|\mathbf{N}(s,t)\|} \, ds \, dt = \iint_{D} \mathbf{F}(\mathbf{X}(s,t)) \cdot \frac{\mathbf{N}(s,t) \cdot \mathbf{N}(s,t)}{\|\mathbf{N}(s,t)\|^2} \, ds \, dt
$$

$$
= \iint_{D} \mathbf{F}(\mathbf{X}(s,t)) \cdot \mathbf{N}(s,t) \, ds \, dt
$$
The vector surface integral of a vector field equals the scalar surface integral of the normal component of the vector field to the surface.

Interpretation: \( \iint_{S} \mathbf{F} \cdot d\mathbf{S} \) measures the vector field flow through the surface. This is called the \textit{flux} of \( \mathbf{F} \) through \( S(0) \).

Compare this to \( \oint_{C} \mathbf{F} \cdot d\mathbf{r} \), the circulation. The vector field flow along \( \gamma(0) \).

Other facts

1. Given a parametrization \( \mathbf{X}: D_{1} \subseteq \mathbb{R}^{2} \rightarrow \mathbb{R}^{2} \) and a \( C^{1} \) 1-1, onto \( \Phi: D_{2} \rightarrow D_{1} \), with inverse \( \Phi^{-1}: D_{1} \rightarrow D_{2} \), a parametrization of \( \mathbf{X} \) is \( \overline{\mathbf{X}}: D_{2} \rightarrow \mathbb{R}^{2} \), where \( \overline{\mathbf{X}} = \mathbf{X} \circ \Phi^{-1} \).
Here $\overrightarrow{x}(s,t) = (\overrightarrow{x} \circ H)(s,t) = \overrightarrow{x}(u(s,t), v(s,t))$.

where $H : D_2 \to D$, $H$ is a bijection map with inverse, and

$H^{-1}(s,t) = (u(s,t), v(s,t))$.
A parametrization is called smooth if both $\mathbf{X}$ and $\mathbf{Y}$ are and if $H$ is $C^1$.

II

Thm. For $f \in C^0$ on a domain including a smooth $\mathbf{X} : D \to \mathbb{R}^3$, then for any smooth parametrization $\mathbf{Y}$ of $\mathbf{X}$,

$$\iint_{\mathbf{X}} f \, ds = \iint_{\mathbf{Y}} f \, ds$$

III

For a curve, an orientation is a choice of continuously varying unit tangent vector along $\mathbf{X}$.

For a surface, an orientation is a choice of continuously varying unit normal vector along $\mathbf{X}$ (above us, below, inside us outside).

orientable

non-orientable
(IV) \[ \text{Then if a reparameterization preserves orientation (i.e., Jacobian(T) > 0 everywhere),} \]
\[ \Rightarrow \int \mathbf{F} \cdot d\mathbf{S} = \int \mathbf{F} \cdot d\mathbf{S} \]
otherwise introduce a minus sign to \( \mathbf{F} \).

Note: Recall \( \mathbf{N}(s,t) = \mathbf{X}_s \times \mathbf{X}_t = -\mathbf{X}_t \times \mathbf{X}_s \).

(V) Orient the surface automatically orient boundary curves on that surface.

Let \( S \) be an oriented surface with boundary in \( \mathbb{R}^3 \). \( \partial S \) is a piecewise \( C^1 \) closed curve.

Let \( \partial p \in \partial S \), where
\[ \partial p = (s_0, t_0) = (x(s_0, t_0), y(s_0, t_0), z(s_0, t_0)) \]
and choose \( \tilde{c} : [a,b] \rightarrow S \subset \mathbb{R}^3 \) a smooth curve such that \( \tilde{c}(a) = \partial p \), and \( \tilde{c} \) ends at \( \partial p' \).
Define \( \overrightarrow{n}(p) = \lim_{t \to 0} \overrightarrow{n}(\overrightarrow{c}(t)) \), and
\[
\overrightarrow{v}(t) = \lim_{t \to 0} \overrightarrow{c}'(t).
\]
Here, \( \overrightarrow{n} \) and \( \overrightarrow{v} \) are based at \( \overrightarrow{p} \) and are perpendicular.
Hence, they determine a 2D subspace containing \( \overrightarrow{v} \) and \( \overrightarrow{n} \).

Then \( \overrightarrow{n} \times \overrightarrow{v} \) is perpendicular to both and using the RHR, determines a unique direction on \( s \).

This is the direction used in Green's Theorem!