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LECTURE 24: DIFFERENTIAL FORMS.

110.211 HONORS MULTIVARIABLE CALCULUS
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Synopsis. A continuation of the last three lectures on differential forms and their structure.

24.1. **A covector product.** Since dx and dy are linear functionals on $\mathbb{R}^2$, viewed as coordinates of $T_{\mathbf{p}}\mathbb{R}^2$ for $\mathbf{p} \in \mathcal{D} \subset \mathbb{R}^2$, they are covectors. And like vectors, beyond summing and constant multiples, one can define products of covectors. But like vectors, products of covectors are not always vectors. Think of the inner, outer, and cross products as forms of multiplication where the output may have the same or a different structure from the inputs. Here, we define a new product on covectors:

Definition 24.1. The *wedge product* of two linear 1-forms ω and ν on $\mathbb{R}^n$ is

$$\omega \wedge \nu(\mathbf{v}_1, \mathbf{v}_2) = \begin{vmatrix} \omega(\mathbf{v}_1) & \omega(\mathbf{v}_2) \\ \nu(\mathbf{v}_1) & \nu(\mathbf{v}_2) \end{vmatrix} = \omega(\mathbf{v}_1)\nu(\mathbf{v}_2) - \omega(\mathbf{v}_2)\nu(\mathbf{v}_1).$$

Exercise 1. Show $\omega \wedge \nu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is linear on each factor, but is not a linear function (it is multi-linear).

Here is a good geometric interpretation: Think of a plane with two vectors

$$\widehat{\mathbf{v}}_1 = \begin{bmatrix} \omega(\mathbf{v}_1) \\ \nu(\mathbf{v}_1) \end{bmatrix}, \quad \text{and} \quad \widehat{\mathbf{v}}_2 = \begin{bmatrix} \omega(\mathbf{v}_2) \\ \nu(\mathbf{v}_2) \end{bmatrix}.$$

Then $\omega \wedge \nu(\mathbf{v}_1, \mathbf{v}_2) = (\widehat{\mathbf{v}}_1 \times \widehat{\mathbf{v}}_2) \cdot \mathbf{k}$ is the *signed area* of the parallelogram in the plane whose sides are $\widehat{\mathbf{v}}_1$ and $\widehat{\mathbf{v}}_2$.

Some notes:

- The wedge product is *skew-symmetric*, or *anti-symmetric*:
$$\nu \wedge \omega(\mathbf{v}_1, \mathbf{v}_2) = -\omega \wedge \nu(\mathbf{v}_1, \mathbf{v}_2).$$
- The wedge product of two 1-forms is also anti-symmetric in its arguments:
$$\begin{aligned} \omega \wedge \nu(\mathbf{v}_2, \mathbf{v}_1) &= \omega(\mathbf{v}_2)\nu(\mathbf{v}_1) - \omega(\mathbf{v}_1)\nu(\mathbf{v}_2) \\ &= -\left(\omega(\mathbf{v}_1)\nu(\mathbf{v}_2) - \omega(\mathbf{v}_2)\nu(\mathbf{v}_1)\right) = -\omega \wedge \nu(\mathbf{v}_1, \mathbf{v}_2). \end{aligned}$$
- $\omega \wedge \nu(\mathbf{v}_1, \mathbf{v}_1) = 0$, always. (Exercise)
- $\omega \wedge \omega(\mathbf{v}_1, \mathbf{v}_2) = 0$, always. (Exercise)
- $(\omega + \mu) \wedge \nu(\mathbf{v}_1, \mathbf{v}_2) = \omega \wedge \nu(\mathbf{v}_1, \mathbf{v}_2) + \mu \wedge \nu(\mathbf{v}_1, \mathbf{v}_2)$. (Exercise)
- The wedge product of two 1-forms is multilinear:
$$\begin{aligned} &\omega \wedge \nu(c_1\mathbf{w}_1 + c_2\mathbf{w}_2, \mathbf{v}) = c_1\omega \wedge \nu(\mathbf{w}_1, \mathbf{v}) + c_2\omega \wedge \nu(\mathbf{w}_2, \mathbf{v}), \quad \text{and} \\ &\omega \wedge \nu(\mathbf{v}, c_1\mathbf{w}_1 + c_2\mathbf{w}_2) = c_1\omega \wedge \nu(\mathbf{v}, \mathbf{w}_1) + c_2\omega \wedge \nu(\mathbf{v}, \mathbf{w}_2). \end{aligned}$$

For 1-forms ω and ν on $\mathbb{R}^n$, $\omega \wedge \nu$ is called a *2-form*, where $\omega \wedge \nu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ acts on paris of vectors. Indeed, let $\omega = \sum a_i \, dx_i$ and $\nu = \sum b_j \, dx_j$ be 2 linear 1-forms. Then

$$\omega \wedge \nu = \sum_{i,j=1}^{n} a_i \, dx_i \wedge b_j \, dx_j = \sum_{i,j=1}^{n} a_i b_j \, dx_i \wedge \, dx_j,$$

since forms are linear on each factor. However, when $i = j$, Definition 24.1 implies

$$dx_i \wedge \, dx_i(\mathbf{v}_1, \mathbf{v}_2) = \begin{vmatrix} dx_i(\mathbf{v}_1) & dx_i(\mathbf{v}_2) \\ dx_i(\mathbf{v}_1) & dx_i(\mathbf{v}_2) \end{vmatrix} = dx_i(\mathbf{v}_1) \, dx_i(\mathbf{v}_2) - \, dx_i(\mathbf{v}_2) \, dx_i(\mathbf{v}_1) = 0.$$

Couple this with the skew-symmetry of the wedge product, so that $dx_i \wedge \, dx_j = - \, dx_j \wedge \, dx_i$, and it becomes obvious that many terms in a generic 2-form can be neglected or combined, simplifying the form.

Exercise 2. Show that, for $\omega = \sum_{i=1}^{n} f_i(x) \, dx_i$ and $\nu = \sum_{j=1}^{n} g_j(x) \, dx_j$, the 2-form $\omega \wedge \nu$ is a sum of, at most, $\binom{n}{2}$ distinct, non-zero terms, which are some function times $dx_i \wedge \, dx_j$, after all simplifications.

Example 24.1. In $\mathbb{R}^3$, with coordinates x, y, and z, let

$$\begin{aligned} \omega &= a_1 \, dx + a_2 \, dy + a_3 \, dz \\ \nu &= b_1 \, dx + b_2 \, dy + b_3 \, dz. \end{aligned}$$

Then

$$\begin{aligned} \omega \wedge \nu &= a_1 b_1 \, dx \wedge \, dx + a_1 b_2 \, dx \wedge \, dy + a_1 b_3 \, dx \wedge \, dz \\ &\quad + a_2 b_1 \, dy \wedge \, dx + a_2 b_2 \, dy \wedge \, dy + a_2 b_3 \, dy \wedge \, dz \\ &\quad + a_3 b_1 \, dz \wedge \, dx + a_3 b_2 \, dz \wedge \, dy + a_3 b_3 \, dz \wedge \, dz \\ &= (a_2 b_3 - a_3 b_2) \, dy \wedge \, dz + (a_3 b_1 - a_1 b_3) \, dz \wedge \, dx + (a_1 b_2 - a_2 b_1) \, dx \wedge \, dy. \end{aligned}$$

Do you recognize the structure of the coefficient (row)-vector here?

Definition 24.2. A *differential 2-form* on a region in $\mathbb{R}^n$ is just a choice of a linear 2-form on each tangent space to the region that varies differentiably with respect to the region points.

For $\omega = \sum_{i=1}^{n} f_i(\mathbf{x}) \, dx_i$ and $\nu = \sum_{j=1}^{n} g_j(\mathbf{x}) \, dx_j$, we have

$$\omega \wedge \nu = \sum_{i,j=1}^{n} h_{ij}(\mathbf{x}) \, dx_i \wedge \, dx_j,$$

with all appropriate cancellations and skew symmetries.

Example 24.2. Let $\omega = x^2y\,dx \wedge dy - xz\,dy \wedge dz$ be a 2-form on $\mathbb{R}^3$, and $\mathbf{p} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Then, at $\mathbf{p}$, we have

$$\begin{aligned} \omega_{\mathbf{p}} &= \left(x^2y\Big|_{\mathbf{p}} \right) dx \wedge dy + \left(xz\Big|_{\mathbf{p}} \right) dy \wedge dz \\ &= 2\,dx \wedge dy - 3\,dy \wedge dz, \end{aligned}$$

a linear 2-form on $T_{\mathbf{p}}\mathbb{R}^3$. Now, if we choose two vectors $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ in $T_{\mathbf{p}}\mathbb{R}^3$, then

$$\begin{aligned} \omega_{\mathbf{p}}(\mathbf{v}_1, \mathbf{v}_2) &= 2\,dx \wedge dy \left(\begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right) - 3\,dy \wedge dz \left(\begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right) \\ &= 2 \begin{vmatrix} 4 & 3 \\ 0 & 1 \end{vmatrix} - 3 \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} = 2(4) - 3(1) = 5. \end{aligned}$$

Note that

$$\begin{vmatrix} 4 & 3 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} dx\left(\begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} \right) & dx\left(\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right) \\ dy\left(\begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} \right) & dy\left(\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right) \end{vmatrix}.$$

Written out, then, a 2-form on $\mathbb{R}^3$, using the coordinates x, y, and z, looks like

$$\mu = f_1(x,y,z)\,dx \wedge dy + f_2(x,y,x)\,dx \wedge dz + f_3(x,y,z)\,dy \wedge dz.$$

And in the case that μ is itself the wedge product of two 1-forms,

$$\omega = g_1\,dx + g_2\,dy + g_3\,dz, \quad \text{and} \quad \nu = h_1\,dx + h_2\,dy + h_3\,dz,$$

then

$$f_1(x,y,z) = g_1(x,y,z)h_2(x,y,z) - g_2(x,y,z)h_1(x,y,z),$$

with the other two coefficients defined similarly.

Notes:

(1) Completely generalizes to $\mathbb{R}^n$, $n > 3$ with a very similar structure.
(2) A generic way to write a differential 2-form on $\mathbb{R}^n$ with coordinates $x_1, \ldots, x_n$ is

$$\mu = \sum_{i,j=1}^{n} f_{ij}\,dx_i \wedge dx_j$$

and leave all cancellations and skew-symmetries up to the reader.

(3) We can continue to construct higher-order forms via the wedge product:
- Let ω_i be a set of m differentiable 1-forms on $\mathbb{R}^n$, for $i = 1, \dots, m$. Then
$$\mu = \omega_1 \wedge \cdots \wedge \omega_m$$
is a differential m-form on $\mathbb{R}^n$ which will ultimately look like
$$\mu = \sum F_{i_1 \cdots i_k}\, dx_{i_1} \wedge \cdots \wedge\, dx_{i_k}$$
with a lot of terms vanishing and other simplifications. At a point $\mathbf{p} \in \mathbb{R}^n$,
$$\mu_{\mathbf{p}} : \underbrace{T_{\mathbf{p}}\mathbb{R}^n \times \cdots \times T_{\mathbf{p}}\mathbb{R}^n}_{m\text{-terms}} \to \mathbb{R},$$
is a linear m-form, and
$$\mu_{\mathbf{p}}(\mathbf{v}_1, \dots, \mathbf{v}_m) = \begin{vmatrix} \omega_1(\mathbf{v}_1) & \cdots & \omega_1(\mathbf{v}_m) \\ \vdots & \ddots & \vdots \\ \omega_m(\mathbf{v}_1) & \cdots & \omega_m(\mathbf{v}_m) \end{vmatrix}.$$
The computation is a very mechanical process.
- Note also that $\mu_{\mathbf{p}}$, a linear m-form, is multilinear, so linear on each argument:
$$\begin{aligned} \mu_{\mathbf{p}}(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, c_1\mathbf{u}_1 + c_2\mathbf{u_2}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_m) \\ &= c_1\mu_{\mathbf{p}}(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{u}_1, \mathbf{v}_{i+1}, \dots, \mathbf{v}_m) \\ &+ c_2\mu_{\mathbf{p}}(\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{u}_2, \mathbf{v}_{i+1}, \dots, \mathbf{v}_m). \end{aligned}$$
- And for ω a differentiable k-form and ν a differentiable ℓ-form, we have $\omega \wedge \nu$ is a differentiable $(k+\ell)$-form

(4) Call a C^1-function f on $\mathbb{R}^n$ a differentiable 0-form. Then, for ω a differentiable k-form,
$$\begin{aligned} f \wedge \omega &= f \cdot \left(\sum F_{i_1 \cdots i_k}\, dx_{i_1} \wedge \cdots \wedge\, dx_{i_k} \right) \\ &= \sum f \cdot F_{i_1 \cdots i_k}\, dx_{i_1} \wedge \cdots \wedge\, dx_{i_k} \end{aligned}$$
is still a k-form.

(5) There exist 2-forms that do not arise as the wedge products of 1-forms.

(6) There are no m-forms on $\mathbb{R}^n$, where $m > n$. Why not?

(7) An n-form on $\mathbb{R}^n$ is also called a *volume form.*

(8) The wedge product is also called the *exterior product* on forms.

Definition 24.3. A differential m-form on $\mathbb{R}^n$, $n \geq m$
$$\mu = \sum_{i_1, \dots, i_m = 1}^{n} F_{i_1 \cdots i_m}\, dx_{i_1} \wedge \cdots \wedge\, dx_{i_m}$$
is a continuous family of linear m-forms $\mu_{\mathbf{p}}$, parameterized by $\mathbf{p} \in \mathbb{R}^n$, such that, at each $\mathbf{p}$,
$$\mu_{\mathbf{p}} : \underbrace{T_{\mathbf{p}}\mathbb{R}^n \times \cdots \times T_{\mathbf{p}}\mathbb{R}^n}_{m\text{-terms}} \to \mathbb{R}$$
is multilinear, so while not a linear function, it is linear on each argument separately.

Here is an alternate view: For each $\mathbf{p} \subset \mathcal{D}\mathbb{R}^n$, and each factor $T_{\mathbf{p}}\mathbb{R}^n$ of $\mu_{\mathbf{p}}$, a choice of $\mathbf{v_p} \in T_{\mathbf{p}}\mathbb{R}^n$ is a vector field on $\mathcal{D} \subset \mathbb{R}^n$. Hence a differential m-form on $\mathcal{D}$ "acts" on a set of m vector fields on $\mathcal{D}$ simultaneously, and returns a function on $\mathcal{D}$.

Example 24.3. Let $\mathbf{F} = 2y\,\mathbf{i} - x\,\mathbf{k}$ be a vector field on $\mathbb{R}^3$, and

$$\omega = x^2y\,dx - x\,dy + y^2z\,dz$$

be a differentiable 1-form. Then, at any given $\mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\mathbf{v} \in \mathbf{F}$, we have

$$\begin{aligned}\omega(\mathbf{v}) &= (x^2y\,dx - x\,dy + y^2z\,dz)(\mathbf{v}) \\ &= x^2y\,dx(\mathbf{v}) - x\,dy(\mathbf{v}) + y^2z\,dz(\mathbf{v}) \\ &= x^2y(2y) - x(0) + y^2z(-x) = 2x^2y^2 - xy^2z.\end{aligned}$$

Hence we can write $\omega(\mathbf{F}) : \mathbb{R}^n \to \mathbb{R}$, as

$$\omega(\mathbf{F})(x, y, x) = 2x^2y^2 - xy^2z.$$

24.2. **Integrating forms.** Yet another alternate view: Forms are "generalized" integrands; One can add them up on each tangent space over a domain. Interpreting them as such and also geometrically will depend on the form and the space we integrate over, though. Here are some examples:

I. Integrating n-forms in $\mathbb{R}^n$:

(a) $n - 1$: Let $f : \mathbb{R} \to \mathbb{R}$ be C^0. Then $\omega = f(x)\,dx$ is a continuous 1-form. For $\mathcal{I} = [a, b] \subset \mathbb{R}$ an interval (already parameterized),

$$\int_{\mathcal{I}} \omega = \int_a^b f(x)\,dx.$$

Any continuous function $f(x)$ on an interval $\mathcal{I} \subset \mathbb{R}$ can be associated to a 1-form in this way, and integrating this form on the interval is performed in the fashion one would employ in first semester, single variable calculus.

(b) If $\omega = f(x, y)\,dx \wedge dy$, on a region $\mathcal{D} \subset \mathbb{R}^2$, then

$$\int_{\mathcal{D}} \omega = \iint_{\mathcal{D}} f(x, y)\,dx\,dy. \tag{24.1}$$

(c) This generalizes quite naturally, and if $\omega = f(\mathbf{x})\,dx_1 \wedge \cdots \wedge dx_n$ is a continuous $n - form$ in $\mathbb{R}^n$, then on some compact, n-dimensional region $\mathcal{R} \subset \mathbb{R}^n$, we can write

$$\int_{\mathcal{R}} \omega = \underbrace{\iint \cdots \int_{\mathcal{R}}}_{n\text{-integrals}} f(mathbfx)\,dx_1 \cdots dx_n. \tag{24.2}$$

This helps to understand the use of the term *volume form* for an n-form in $\mathbb{R}^n$; If $f(\mathbf{x}) \equiv 1$, then integrating ω over $\mathcal{R}$ yields the volume of $\mathcal{R}$:

- For a form defined on a region, we always integrate the form over a subset of that region of the same "size" as that of the order of the form.

- Notice that in both Eqns 24.1 and 24.2, we use only a single integral sign, even for multiple integrals. The form and the region identify the type of integration, so we do not need to use multiple integrals. This will be very useful later.

I. Integrating m-forms in $\mathbb{R}^n$, $m < n$: Here, we highlight, via a few examples, the integration of forms on spaces and see how one can interpret these quantities in ways that we have already developed and discussed, but using this new language of forms:

Example 24.4. ***Integrating 1-forms and the circulation of a vector field.*** Let $m = 1$. Then, on a curve $\mathbf{c} : [a, b] \to \mathbb{R}^n$, with

$$\omega = f_1(\mathbf{x})\, dx_1 + \ldots + f_n(\mathbf{x})\, dx_n = \sum_{i=1}^{n} f_i(\mathbf{x})\, dx_i,$$

we can write $\omega = \mathbf{F} \bullet d\mathbf{s}$, where $\mathbf{F}(\mathbf{x}) = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$ can be interpreted as a vector field, and $d\mathbf{s} = \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix}$ is a vector form of infinitesimal displacement. Then we have

$$\begin{aligned}
\int_{\mathbf{c}} \omega = \int_{\mathbf{c}} \mathbf{F} \bullet d\mathbf{s} &= \int_{\mathbf{c}} \omega_{\mathbf{c}} \left(\mathbf{c}(t)\right)\, dt \\
&= \int_a^b \mathbf{F}\left(\mathbf{c}(t)\right) \bullet d\mathbf{s}\left(\mathbf{c}'(t)\right) \\
&= \int_a^b \mathbf{F}\left(\mathbf{c}(t)\right) \bullet \begin{bmatrix} dx_1\left(\mathbf{c}'(t)\right) \\ \vdots \\ dx_n\left(\mathbf{c}'(t)\right) \end{bmatrix} dt \\
&= \int_a^b \mathbf{F}\left(\mathbf{c}(t)\right) \bullet \mathbf{c}'(t)\, dt.
\end{aligned}$$

Hence our interpretation is that integrating a 1-form over a curve in $\mathbb{R}^n$ is the same as calculating the circulation of a vector field over the curve (a vector line integral of the field), when the vector field is the coefficient vector of the form. In this interpretation, we have

$$d\mathbf{s} = \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix} = \begin{bmatrix} x_1'(t)\, dt \\ \vdots \\ x_n'(t)\, dt \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} dt = \mathbf{c}'(t)\, dt.$$

Example 24.5. ***Integrating 2-forms and surface integrals.*** For $m = 2$. Then, on a surface $\mathbf{X} : \mathcal{D} \subset \mathbb{R}^2 \to \mathbb{R}^n$, with

$$\omega = \sum_{i,j=1}^{n} F_{ij}(\mathbf{x})\, dx_i \wedge dx_j, \text{(neglecting simplifications)}$$

we have, with $\mathbf{X}(\mathcal{D}) = \mathcal{R} \subset \mathbb{R}^n$,

$$\int_{\mathcal{R}} \omega = \int_{\mathbf{X}} \omega_{\mathbf{X}} \left(\mathbf{X}_s, \mathbf{X}_t\right) \, ds \wedge \, dt,$$

where

$$\mathbf{X}_s = \begin{bmatrix} \frac{\partial x_1}{\partial s}(s,t) \\ \vdots \\ \frac{\partial x_1}{\partial s}(s,t) \end{bmatrix}, \quad \text{and} \quad \mathbf{X}_t = \begin{bmatrix} \frac{\partial x_1}{\partial t}(s,t) \\ \vdots \\ \frac{\partial x_1}{\partial t}(s,t) \end{bmatrix}$$

are the partial derivative vectors determined by the parameterization $\mathbf{X}(s,t) = (x_1(s,t), \ldots, x_n(s,t))$. Thus, we have

$$\int_{\mathcal{R}} \omega = \int_{\mathbf{X}} \omega_{\mathbf{X}} \left(\mathbf{X}_s, \mathbf{X}_t\right) \, ds \wedge \, dt = \iint_{\mathcal{R}} \mathbf{F} \bullet \, d\mathbf{S},$$

which looks something like a vector surface integral, but where *we need to understand and interpret these pieces appropriately.* Here $\mathbf{F}$ is a vector of all of the $\mathbf{F}_{ij}$s, but is not a vector field in the sense that we have already defined it. For evidence of this, there will be $\binom{n}{2}$ elements in $\mathbf{F}$ after simplifications. This is typically too many to actually serve as a vector field. Also, the vector $d\mathbf{S}$ is a vector of the associated term $dx_i \wedge dx_j$, and is also a vector of size $\binom{n}{2}$, too many to serve geometrically as infinitesimal displacement like it did in $\mathbb{R}^3$. However, at least formally, we can write

$$\int_{\mathbf{X}} \omega_{\mathbf{X}} \left(\mathbf{X}_s, \mathbf{X}_t\right) \, ds \wedge \, dt = \int_{\mathcal{D}} \mathbf{F}\left(\mathbf{X}(s,t)\right) \bullet \, d\mathbf{S}\left(\mathbf{X}_s, \mathbf{X}_t\right),$$

where, for each term in $d\mathbf{S}$, we have

$$\begin{aligned} dx_i \wedge \, dx_j \left(\mathbf{X}_s, \mathbf{X}_t\right) &= \begin{vmatrix} dx_i\left(\mathbf{X}_s\right) & dx_i\left(\mathbf{X}_t\right) \\ dx_j\left(\mathbf{X}_s\right) & dx_j\left(\mathbf{X}_t\right) \end{vmatrix} ds \wedge \, dt = \begin{vmatrix} \frac{\partial x_i}{\partial s} & \frac{\partial x_i}{\partial t} \\ \frac{\partial x_j}{\partial s} & \frac{\partial x_j}{\partial t} \end{vmatrix} ds \wedge \, dt \\ &= \frac{\partial(x_i, x_j)}{\partial(s,t)} \, ds \wedge \, dt. \end{aligned}$$

For instance, if the surface $\mathbf{X} : \mathcal{D} \to \mathbb{R}^4$ with coordinates (x, y, z, u), then the 2-form ω can look like

$$\omega = F_{12} \, dx \wedge \, dy + F_{13} \, dx \wedge \, dz + F_{14} \, dx \wedge \, du + F_{23} \, dy \wedge \, dz + F_{24} \, dy \wedge \, du + F_{34} \, dz \wedge \, du.$$

Then we can write

$$\mathbf{F} = \begin{bmatrix} F_{12}(x,y,z,u) \\ F_{13}(x,y,z,u) \\ \vdots \\ F_{34}(x,y,z,u) \end{bmatrix}, \quad \text{and} \quad \mathbf{dS} = \begin{bmatrix} dx \wedge \, dy \\ dx \wedge \, dz \\ \vdots \\ dz \wedge \, du \end{bmatrix} = \begin{bmatrix} \frac{\partial(x,y)}{\partial(s,t)} \\ \frac{\partial(x,z)}{\partial(s,t)} \\ \vdots \\ \frac{\partial(z,u)}{\partial(s,t)} \end{bmatrix} ds \wedge \, dt,$$

even as these $6 = \binom{4}{2}$-vectors do not correspond to geometric objects on $\mathbb{R}^4$.

Example 24.6. ***Integrating 2-forms in $\mathbb{R}^3$ and the flux of a vector field.*** In the special case where the surface in in $\mathbb{R}^3$, so $n = 3$ and $m = 2$, we do have a good

geometric interpretation of the integral of a 2-form on the surface. For a parameterized surface $\mathcal{S} \subset \mathbb{R}^3$, with $\mathbf{X} : \mathcal{D} \subset \mathbb{R}^2 \to \mathbb{R}^3$, and $\mathcal{S} = \mathbf{X}(\mathcal{D})$, we have ω with only $3 = \binom{3}{2}$ terms, so

$$\omega = F_{12}(\mathbf{x})\, dx \wedge dy + F_{13}(\mathbf{x})\, dz \wedge dx + F_{23}(\mathbf{x})\, dx \wedge dy.$$

Then we have

$$\int_{\mathcal{S}} \omega = \int_{\mathcal{D}} \omega_{\mathbf{X}} \left(\mathbf{X}_s, \mathbf{X}_t\right) ds \wedge dt = \int_{\mathcal{S}} \mathbf{F} \bullet d\mathbf{S}.$$

where $\mathbf{F}(\mathbf{x}) = F_{12}(\mathbf{x})\,\mathbf{i} + F_{13}(\mathbf{x})\,\mathbf{j} + F_{23}(\mathbf{x})\,\mathbf{k}$ is an actual vector field on $\mathbb{R}^3$ (it has the correct dimension), and

$$d\mathbf{S} = \begin{bmatrix} dy \wedge dz \\ dz \wedge dx \\ dx \wedge dy \end{bmatrix} = \begin{bmatrix} \frac{\partial(y,z)}{\partial(s,t)} \\ \frac{\partial(z,x)}{\partial(s,t)} \\ \frac{\partial(x,y)}{\partial(s,t)} \end{bmatrix} ds \wedge dt.$$

Hence we can interpret the integration of a 2-form on a surface in $\mathbb{R}^3$ as the calculation of the flux of a vector field through a surface, where the vector field is the coefficient vector of the 2-form. This is a <u>*vector surface integral*</u> of the field.

Special note here: Notice how we defined the "middle" term in ω here using $dz \wedge dx$ instead of $dx \wedge dz$. Geometrically speaking, there is a reason for this, and we will get to that with a little more structure in the next (and final) lecture. But for now, to associate the three functions F_{12}, F_{13}, and F_{23} with the actual components of a vector field $\mathbf{F}$, we need to address an issue of a minus sign introduced in this middle term. We choose to respect the minus sign by reversing the terms in the middle wedge. Keep track of this and hold your thoughts for now.

More generally, let $\mathcal{R} \subset \mathbb{R}^n$ be an m-dimensional region parameterized by $\mathbf{X} : \mathcal{D} \subset \mathbb{R}^m \to \mathbb{R}^n$, where $\mathbf{X}(\mathcal{D}) = \mathcal{R}$. Then, for ω a differential m-form on $\mathbb{R}^n$,

$$\omega\Big|_{\mathbf{X}(\mathcal{D})} = \omega_{\mathbf{X}(\mathcal{D})} = \omega_{\mathbf{X}}$$

is an m-form on $\mathcal{R}$ which can be expressed and integrated via the parameterization. Indeed, with the coordinates $(s, t, \ldots, u)$ for $\mathcal{D} \in \mathbb{R}^m$, we have

$$\begin{aligned} \int_{\mathcal{R}} \omega &= \int_{\mathbf{X}} \omega_{\mathbf{X}} \left(\mathbf{X}_s, \mathbf{X}_t, \ldots, \mathbf{X}_u\right) ds \wedge dt \wedge \cdots \wedge du \\ &= \int_{\mathcal{R}} \mathbf{F} \bullet d\mathbf{S}, \end{aligned}$$

where the vector $\mathbf{F}$ of all of the form coefficient functions $F_{i_1 i_2 \cdots i_m}$, and the vector $d\mathbf{S}$, containing all of the respective form pieces $dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m}$, will each have only $\binom{n}{m}$-distinct elements. They will not generally correspond to vector fields in the way we define them (there are often too many elements). We will go any deeper than this here, but do note the following: Things tend to line up well, combinatorially, when one integrates a 1-form over a curve (the number of elements if $\mathbf{F}$ and $d\mathbf{s}$ is $\binom{n}{1} = n$), and when integrating an $(n-1)$-form over a hypersurface (of dimension again $(n-1)$) since

then, again, the constituents vectors $\mathbf{F}$ and $d\mathbf{S}$ have size $\binom{n}{n-1} = n$ each. At that point, we can again think of $\mathbf{F}$ as a vector field.

Exercise 3. Let $\mathbf{F} = \begin{bmatrix} y \\ x \end{bmatrix}$ a vector field on $\mathbb{R}^2$ and $ds = \begin{bmatrix} dx \\ dy \end{bmatrix}$, the quantity $\omega = \mathbf{F} \bullet d\mathbf{s}$ is the 1-form $\omega = y\,dx + x\,dy$. Calculate $\int_{\mathbf{c}} \omega$, where $\mathbf{c} : [0,2] \to \mathbb{R}^2$ is defined by $\mathbf{c}(t) = \begin{bmatrix} t^2 \\ t^3 \end{bmatrix}$. (Hint: The answer is 32.)

And now we end this lecture with some examples of techniques and quantities you are already familiar with, but now revisited in the new language of forms.

Example 24.7. ***Integrating a 1-form in $\mathbb{R}$ and the Substitution Method in Calculus I.*** Let $\omega = f(u)\,du$, a differential 1-form on $\mathcal{I} = [c,d]$. Then $\displaystyle\int_{\mathcal{I}} \omega = \int_a^b f(u)\,du$ like in Calculus I. But let's reparameterize $\mathcal{I}$ via the function

$$g : \mathcal{J} \to \mathcal{I}, \quad g : [a,b] \to [c,d]$$

so that $u = g(x)$, and $c = u(a)$ and $d = u(b)$.

Now using the reparameterization, we get

$$\int_{\mathcal{I}} \omega = \int_{\mathcal{J}} \omega_g\left(g'(x)\right)\,dx = \int_a^b f\left(g(x)\right) \cdot g'(x)\,dx,$$

so that

$$\int_{c=g(a)}^{d=g(b)} f(u)\,du = \int_a^b f\left(g(x)\right) \cdot g'(x)\,dx.$$

Do you remember the structure of the *Substitution Method* in first semester calculus?

Example 24.8. ***Fubini and the Change of Variables Theorem in the language of forms.*** Here is a curious and beautiful fact: Recall that forms are skew-symmetric, so $dx \wedge dy = -dy \wedge dx$. Hence for $\omega = f(x,y)\,dx \wedge dy$, we have $-\omega = f(x,y)\,dy \wedge dx$. Let $\mathcal{R} \subset \mathbb{R}^2$ be rectangular. Then, by Fubini's Theorem,

$$\iint_{\mathcal{R}} f(x,y)\,dx\,dy = \iint_{\mathcal{R}} f(x,y)\,dy\,dx.$$

However,

$$\int_{\mathcal{R}} \omega = \iint_{\mathcal{R}} f(x,y)\,dx \wedge dy \stackrel{?}{=} \iint_{\mathcal{R}} f(x,y)\,dy \wedge dx = -\int_{\mathcal{R}} \omega.$$

What is going on here? Actually, nothing out of the ordinary. Switching the order of integration is like a reparameteriztion of the plane, from the xy-plane to the yx-plane, and the switching function is $T : (y,x) = (x,y)$. This, in fact, is an orientation-reversing reparameterization, since

$$\mathbf{Jac}(T) = \begin{vmatrix} \frac{\partial}{\partial y}[x] & \frac{\partial}{\partial y}[y] \\ \frac{\partial}{\partial x}[x] & \frac{\partial}{\partial x}[y] \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

Now the standard Change of Variables formula is

$$\iint_{\mathcal{R}} f(x,y)\,dx\,dy = \iint_{\mathcal{R}} f(x,y)\left|\frac{\partial(x,y)}{\partial(y,x)}\right|\,dy\,dx$$

with the *absolute value* of the Jacobian determinant. But the absolute value is artificial, and is a convenient shortcut to mask a much deeper structure.

Indeed, this curious fact relies on the idea that for Fubini's Theorem, we avoid orientation and forms, and simply state that the order of integration (over a rectangle) does not matter. But with forms, it does matter, as orientation is critical. In the above case, the change in orientation due to the reparameterization introduces a minus sign. But the switch from the form $dx \wedge dy$ to $dy \wedge dx$ introduces another, which conveniently cancels out the former. Simply changing the order of integration and taking the absolute value of the Jacobian works. Something to think about. Without forms (and orientation), we can conveniently simply change the order of integration in Fubini's Theorem without violating rules regarding orientation of variable changes.

Example 24.9. ***Integrating a 2-form on a surface in three space.*** Let

$$\mathcal{M} = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; z = \sqrt{1 - x^2 - y^2} \right\}$$

be the unit sphere above the xy-plane, and $\omega = z^2\, dx \wedge dy$ be a differentiable 2-form on $\mathbb{R}^3$. Evaluate $\int_{\mathcal{M}} \omega$.

Strategy. Parameterize the hemisphere $\mathcal{M}$ and calculate the integral via the parameterization.

Solution. Use the function

$$\mathbf{X}(r, \theta) = \left(r\cos\theta, r\sin\theta, \sqrt{1 - r^2} \right)$$

so that the paramter region is the rectangle $\mathcal{D} = [0, 1] \times [0, 2\pi]$ in the $r\theta$-plane, as shown in Figure 24.1 below:

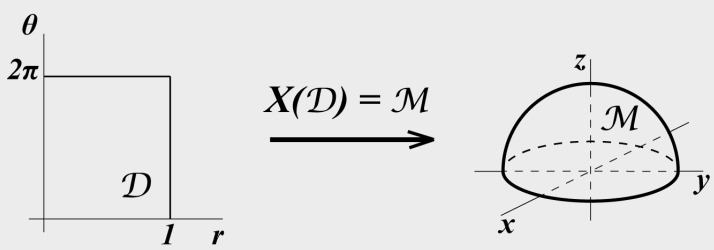

FIGURE 24.1. A parameterization of the unit northern hemisphere in $\mathbb{R}^3$.

Then we can calculate using the parameterization:

$$\begin{aligned}
\int_{\mathcal{M}} \omega &= \int_{\mathcal{D}} \omega_{\mathbf{X}(r,\theta)} \left(\frac{\partial X}{\partial r}(r, \theta), \frac{\partial X}{\partial \theta}(r, \theta) \right) dr \wedge d\theta \\
&= \int_{\mathcal{D}} \omega_{\mathbf{X}(r,\theta)} \left(\begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{-r}{\sqrt{1-r^2}} \end{bmatrix}, \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix} \right) dr \wedge d\theta.
\end{aligned}$$

Here, think of these as part of a cylindrical coordinate system on $\mathbb{R}^3$, with the last coordinate $z = 1 - r^2$. Then continuing:

$$\begin{aligned}
\int_{\mathcal{M}} \omega &= \int_{\mathcal{D}} (1 - r^2) \begin{vmatrix} dx\left(\begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{-r}{\sqrt{1-r^2}} \end{bmatrix} \right) & dx\left(\begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix} \right) \\ dy\left(\begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{-r}{\sqrt{1-r^2}} \end{bmatrix} \right) & dy\left(\begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix} \right) \end{vmatrix} dr \wedge d\theta \\
&= \int_{\mathcal{D}} (1 - r^2) \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr \wedge d\theta = \int_{\mathcal{D}} (1 - r^2) \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} dr \wedge d\theta \\
&= \int_{\mathcal{D}} (1 - r^2) r\, dr \wedge d\theta = \int_0^{2\pi} \int_0^1 (r - r^3)\, dr\, d\theta \\
&= \int_0^{2\pi} \left[\left(\frac{r^2}{2} - \frac{r^4}{4} \right) \Big|_0^1 \right] d\theta = \frac{\theta}{4}\Big|_0^{2\pi} = \frac{\pi}{2}.
\end{aligned}$$