Synopsis. Today we begin the study of Chapter 3 on vector-valued functions. For the most part, there are only two topics of discussion here: paths or curves and vector fields, respectively defined as functions from the real line into n-space, or functions from n-space into itself. The reason for an entire chapter on these two items is that they play a huge role in a solid general understanding of all of the calculus of vector-valued functions of more than one variable. They also introduce the idea of a geometric object begin completely defined by a function, allowing us to fold geometry into the analysis of functions in a fundamental way. This is one of the core principles of higher mathematics. Today, curves in n-space and some of their properties. One defining characteristic of a curve in n-space is that its length should be independent of its parameterization, even though we calculate the length using the parameterization. This extra document details why this is so:

Helpful Documents. PDF: ParameterizationIndependence.

Curves in \( \mathbb{R}^n \). We start with a definition:

**Definition 9.1.** A *curve* or *path* in \( \mathbb{R}^n \) is a continuous function \( x : I \subset \mathbb{R} \rightarrow \mathbb{R}^n \), where

\[
x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \text{ defined on an interval.}
\]

Note that the image of \( x(t) \) is an \( n \)-vector for each value of \( t \in I \). If \( x \) is differentiable as a function, then its derivative is also an \( n \)-vector, and

\[
\frac{d}{dt} x(t) = x'(t) = \begin{bmatrix} \frac{dx_1}{dt}(t) \\ \vdots \\ \frac{dx_n}{dt}(t) \end{bmatrix} = \begin{bmatrix} x'_1(t) \\ \vdots \\ x'_n(t) \end{bmatrix}.
\]

We sometimes call the derivative vector the *velocity*, and denote it \( v(t) = x'(t) \).

At a point \( t_0 \in I \), \( v(t_0) \) is represented by a vector, based at \( x(t_0) \) and tangent to the curve \text{image}(x)\) Here, we simply denote the entire curve as \( x \). Now, as long as \( v(t_0) \neq 0 \), this vector defines a unique tangent line to \( x \) at \( t = t_0 \), parameterized as

\[
\ell(s) = x(t_0) + sv(t_0), \quad \text{or} \quad \ell(s) = x(t_0) + (t - t_0)v(t_0), \quad \text{for} \quad s = t - t_0.
\]

Note that the line \( \ell = \text{span} \{ v(t_0) \} \).

Here, the *speed* of \( x(t) \) at \( t = t_0 \) is simply the size of the velocity vector at \( t_0 \), so \( ||v(t_0)|| \). The interpretation is of a bead moving along a piece of wire that is the curve. The bead is at \( x(t_0) \) at time \( t = t_0 \) and moving with (instantaneous) speed \( ||v(t_0)|| \) then. All of this is a topic of a standard single variable calculus course, since all of the derivatives here are
calculated according to the component functions \( x_i : I \to \mathbb{R} \), each of which is a real-valued on \( I \subset \mathbb{R} \).

Indeed, let \( x = f(t) \) and \( y = g(t) \), for \( t \in I \subset \mathbb{R} \), define a parametric curve in \( \mathbb{R}^2 \). If \( f, g \in C^1 \), then \( \frac{dx}{dt} = f'(t) \) and \( \frac{dy}{dt} = g'(t) \), and when defined,

\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}
\]
defines the tangent line in \( \mathbb{R}^2 \) to the curve at \( (x_0, y_0) \):

\[
y = \left( \frac{dy}{dx} \right)_{(x_0, y_0)} (x - x_0) + y_0, \quad x_0 = f(t_0), \text{ and } y_0 = g(t_0).
\]

This construction was useful for studying curves that are defined only implicitly and not representable as functions \( y(x) \) or \( x(y) \): If a curve is defined as \( F(x, y) = 0 \), then we can calculate \( \frac{dy}{dx} \) in two ways: (1) implicitly, or (2) via a parameterization like above. But we can use the language of vector calculus, now, to revisit these methods:

**Implicit differentiation.** Assume that \( y = y(x) \) is an implicit function of \( x \). The equation \( F(x, y) = 0 \) looks like \( F(x, y(x)) = 0 \), and is only a function of \( x \). Thus we can differentiate with respect to \( x \) and get

\[
\frac{d}{dx}F(x, y(x)) = \frac{\partial}{\partial x}F(x, y) + \frac{\partial}{\partial y}F(x, y) \frac{dy}{dx} = 0.
\]

Thus, we get

\[
\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}.
\]

**Via parameterization.** Both \( x = x(t) \) and \( y = y(t) \) are functions of \( t \), so \( F(x, y) = F(x(t), y(t)) = 0 \), and \( F \) is only a function of \( t \). Thus

\[
\frac{d}{dt}F(x(t), y(t)) = F_x(x, y) \frac{dx}{dt} + F_y(x, y) \frac{dy}{dt} = 0.
\]

This implies again the SAME result:

\[
\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}.
\]

Thinking of a curve as a function affords us all of the tools of calculus to study the geometry of curves:

1. We can attribute higher derivatives to geometric features like acceleration,

\[
a(t) = \frac{d}{dt}v(t) = \frac{d^2}{dt^2}x(t),
\]

and jerk, etc.
(2) We can recover quantities like distance via integrating velocity, so that 
\[ x(t) = \int_{t_0}^{t} v(s) \, ds. \]

Do keep in mind, though, that integrating a vector means integrating each component, noting that the constant of integration is, again, a vector.

(3) Derivative rules, again, behave well with respect to curves. For example, the Product Rule and the Dot Product:
\[ \frac{d}{dt}[x \cdot y(t)] = \frac{dx}{dt} \cdot y(t) + x(t) \cdot \frac{dy}{dt}. \]

(4) Facilitates geometric study:

**Example 9.2.** If \( x(t) \subset \mathbb{R}^n \) is a \( C^1 \)-curve, with \( ||x(t)|| = c > 0 \), for all \( t \in I \), then \( x'(t) \cdot x(t) = 0 \), for every \( t \in I \).

**Exercise 1.** Prove this result.

Recall from Calculus II, for \( f : [\alpha, \beta] \to \mathbb{R} \), the length of \( \text{graph}(f) \subset \mathbb{R}^2 \) on \( [\alpha, \beta] \) is
\[ L = \int_{\alpha}^{\beta} \sqrt{1 + (f'(x))^2} \, dx, \]
or if the curve is a parametric curve,
\[ L = \int_{\alpha}^{\beta} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt. \]

**Exercise 2.** Show that these two quantities are the same.

This last formula is tailor-made for us: Let \( x : [a, b] \to \mathbb{R}^2 \), \( x(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \). Here, given a partition on the interval \([a, b]\),
\[ a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b, \]
one looks for the approximate length of the curve on the subinterval \([t_{i-1}, t_i]\), and then adds up the approximations on each subinterval to get an approximation of the length of the curve. For each subinterval \([t_{i-1}, t_i]\), calculate \( \Delta t_i \). Now approximate the length of the curve on a subinterval by using Euclidean distance between \( x(t_{i-1}) \) and \( x(t_i) \). The approximate length of the curve in the \( i \)th subinterval is
\[ ||x(t_i) - x(t_{i-1})|| = \sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2}. \]

But we can write
\begin{align*}
  x(t_i) - x(t_{i-1}) &= \Delta x_i = x'(t^*_i) \Delta t_i \\
  y(t_i) - y(t_{i-1}) &= \Delta y_i = y'(t^{**}_i) \Delta t_i
\end{align*}
by the Mean Value Theorem for some \( t^*_i \) and \( t^{**}_i \) in \([t_{i-1}, t_i]\). So the approximate length of the curve, given the partition, is
\[ \text{approx } L = \sum_{i=1}^{n} \sqrt{(x'(t^*_i))^2 + (y'(t^{**}_i))^2} \, \Delta t_i. \]
And the actual length is found by taking the limit as the largest \( \Delta t_i \to 0 \):

\[
L = \lim_{\max_i, \Delta t_i \to 0} \sum_{i=1}^{n} \sqrt{(x'(t_i^*)^2 + (y'(t_i^*)^2)} \Delta t_i
\]

\[
= \int_{a}^{b} \sqrt{(x'(t))^2 + (y'(t))^2} \, dt
\]

\[
= \int_{a}^{b} ||x'(t)|| \, dt,
\]

where the quantity \( ||x'(t)|| \) is the size of the velocity vector at time \( t \), otherwise known as the speed of the curve at \( t \). All of this works in \( \mathbb{R}^n \), \( n \in \mathbb{N} \).

**Definition 9.3.** The length of \( x : [a, b] \to \mathbb{R}^n \), a \( C^1 \)-parameterized curve in \( \mathbb{R}^n \) is

\[
L(x) = \int_{a}^{b} ||x'(t)|| \, dt.
\]

Some notes:

- One integrates speed to recover distance (length traveled)!
- Even if the curve is only piecewise \( C^1 \) (so maybe it has corners), this still works, as integrals are additive.
- This formula seems to critically depend on the parameterization. But it does not! (See the Helpful document for a proof.) To verify, reparameterize and reintegrate, or better yet, parameterize intrinsically, using length itself as the parameter on the curve.

Let \( x : [a, b] \to \mathbb{R}^n \) be a path with non-zero velocity everywhere (so that \( v(t) \neq 0 \), \( \forall t \in [a, b] \)). Denote by \( p_0 = x(a) \), and \( p = x(s) \), where

\[
s(t) = \int_{a}^{t} ||x'(\tau)|| \, d\tau.
\]

(Note that the use of \( \tau \) is simply a dummy variable in a definite integral, and never actually appears on the curve.) Some things to think about:

- Since \( x'(t) \neq 0 \), the length is always positive, and \( s(t) \) is a strictly increasing function. As such, it is invertible, and we can reparameterize \( x(t) \) to

\[
x(s) = x(t(s))
\]

as a function of \( s \).
- In practice, \( t(s) \) may be difficult of near impossible to find, but the total length of the curve is

\[
s(b) = \int_{a}^{t} ||x'(\tau)|| \, d\tau = \int_{a}^{t} ||x'(t)|| \, dt,
\]

which is just the length of the curve in the \( t \) parameter. Hence reparameterization does not change length.
- \( s(t) \) is \( C^1 \) when \( x \) is, and

\[
s'(t) = \frac{ds}{dt} = \frac{d}{dt} \left[ \int_{a}^{t} ||x'(\tau)|| \, d\tau \right] = ||x'(t)||.
\]
So under this reparameterization, the derivative is just the spread of the curve at the old value of $t$.

So we can use this to calculate the tangent vector in the new parameter: Write $\mathbf{x}(t) = \mathbf{x}(s(t))$. The differentiate, using the Chain Rule:

$$\mathbf{x}'(t) = \frac{d}{dt} \mathbf{x}(s(t)) = \mathbf{x}'(s) \cdot s'(t) = \mathbf{x}(s) \|\mathbf{x}'(t)\|, \text{ so } \mathbf{x}'(s) = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}.$$ 

Conclusions?

(1) In the new parameter, the arclength traverses the curve at unit speed always.
(2) $\mathbf{x}'(s)$ is just the normalization of the tangent vector at the same point as $\mathbf{x}(t)$.

**Definition 9.4.** For a $C^1$-path $\mathbf{x} : [a, b] \rightarrow \mathbb{R}^n$, the *unit tangent vector* to $\mathbf{x}$ at $t = t_0$ is

$$T(t_0) = \frac{\mathbf{x}'(t_0)}{\|\mathbf{x}'(t_0)\|},$$

and is just the normalized velocity.

This concept of a normalized velocity vector will be very important later on in the course.