HOMEWORK PROBLEM SET 3: DUE FEBRUARY 20, 2019

AS.110.211 HONORS MULTIVARIABLE CALCULUS
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Question 1. For the functions $\mathbf{f}$ and points $\mathbf{a}$ indicated, calculate $D\mathbf{f}(\mathbf{a})$:

(a) $\mathbf{f}(x, y, z) = \big(2x - 3y + 5z, x^2 + y, \ln(yz)\big)$, and $\mathbf{a} = (3, -1, -2)$.

(b) $\mathbf{f}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2y \\ x + y^2 \\ \cos \pi xy \end{bmatrix}$, and $\mathbf{a} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Question 2. Explain fully why the function $\mathbf{f}(x, y) = \left(\dfrac{xy^2}{x^2 + y^2}, \dfrac{x}{y} + \dfrac{y}{x}\right)$ is differentiable at every point of its domain.

Question 3. Find equations for the following spaces:

(a) All planes tangent to $z = x^2 - 6x + y^3$ that are parallel to the plane $4x - 12y + z = 7$.

(b) The hyperplane tangent to the 4-dimensional paraboloid $x_5 = 10 - \big(x_1^2 + 3x_2^2 + 2x_3^2 + x_4^2\big)$ at the point $(2, -1, 1, 3, -8)$.

Question 4. Let $g(x, y) = \sqrt[3]{xy}$, do the following:

(a) Determine if g is continuous at $(0, 0)$.
(b) Calculate the partials of g, when $xy \neq 0$.
(c) Show that $g_x(0, 0)$ and $g_y(0, 0)$ exist by finding the limits.
(d) Determine if the partials are continuous at $(0, 0)$.
(e) Determine if the graph of g has a tangent plane at $(0, 0)$. It is not necessary, but if you want, show the graph of g.
(f) Determine if g is differentiable at $(0, 0)$.

Question 5. Do the following:

(a) Verify the Sum Rule for derivatives when $\mathbf{f}(x, y, z) = (xyz^2, xe^{-y}, y \sin xz)$ and $\mathbf{g}(x, y, z) = (x - y, x^2 + y^2 + z^2, \ln(xz + 2))$.
(b) Verify the Product and Quotient Rules for derivatives when $f(x, y) = x^2y + y^3$ and $g(x, y) = \frac{x}{y}$.

(c) Verify that the Product Rule holds for the derivative of a cross product in $\mathbb{R}^3$. That is, directly calculate $D\mathbf{h}(\mathbf{x})$, for $\mathbf{h} : \mathbb{R}^3 \to \mathbb{R}^3$, $\mathbf{h}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \times \mathbf{g}(\mathbf{x})$, where $\mathbf{f} : \mathbb{R} \to \mathbb{R}^3$ and $\mathbf{g} : \mathbb{R} \to \mathbb{R}^3$ are C^1 functions, and show that it equals the Product Rule $D\mathbf{h}(\mathbf{x}) = D\mathbf{f}(\mathbf{x}) \times \mathbf{g}(\mathbf{x}) + \mathbf{f}(\mathbf{x}) \times D\mathbf{g}(\mathbf{x})$.

Question 6. For the function $F(x, y, z) = 2x^3y + xz^2 + y^3z^5 - 7xyz$, do the following:

(a) Calculate F_{xx}, F_{yy}, F_{zz}.

(b) Calculate all mixed second-partials and verify in all cases that the order in taking partials does not matter.

(c) Determine if $F_{xyx} = F_{xxy}$. If so, use the mixed partials theorem to show that this must be the case.

(d) Do the same for F_{xyz} and F_{yzx}.

Question 7. Let $f(x, y, z) = \ln\left(\frac{xy}{z}\right)$. Give general formulas for $\frac{\partial^n f}{\partial x^n}$, $\frac{\partial^n f}{\partial y^n}$, and $\frac{\partial^n f}{\partial z^n}$ where $n \geq 1$. What can you say about the mixed partials?

Question 8. Let

$$f(x, y) = \begin{cases} xy\left(\frac{x^2-y^2}{x^2+y^2}\right) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

(a) Find $f_x(x, y)$ and $f_y(x, y)$ for $(x, y) \neq (0, 0)$.

(b) Either by hand (using limits), or by using part **(a)**, find the partial derivatives $f_x(0, y)$ and $f_y(x, 0)$.

(c) Find the values to $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$. Reconcile this with Theorem 4.3 in the text on page 137, on the equality of mixed partials.

Question 9. Do the following:

(a) If $f(x, y) = \sin(xy)$ and $x = s + t$ and $y = s^2 + t^2$, find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$, both by direct substitution, and by means of the Chain Rule.

(b) Suppose that $z = x^2 + y^3$, where $x = uv$ and y is also some function of u and v. Suppose further that, when $(u, v) = (2, 1)$, then $\frac{\partial y}{\partial v} = 0$. Determine $\frac{\partial z}{\partial v}(2, 1)$.