HOMEWORK PROBLEM SET 4: DUE FEBRUARY 27, 2019

AS.110.211 HONORS MULTIVARIABLE CALCULUS
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Question 1. Do the following:

(a) For the C^1 function $z = f(x, y)$, let $x = 2uv$ and $y = u^2 + v^2$. Show that

$$\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} = 2x\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] + 4y\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}.$$

(b) For the C^1 function $w = g(u)$, where $u = \frac{x^2-y^2}{x^2+y^2}$, show

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 0.$$

Question 2. For $\mathbf{f}(x) = (x^2, \cos 3x, \ln x)$, and $g(s, t, u) = s + t^2 + u^3$, calculate $D(\mathbf{f} \circ g)$ in two ways: (1) by first writing an expression for $(\mathbf{f} \circ g)$ and then calulating the derivative, and (2) by using the Chain Rule with the derivatives $D\mathbf{f}$ and Dg.

Question 3. Using Example 6 on pages 152-3 of the text, determine the operators $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial y^2}$ in terms of the polar coordinate partial derivative operators $\frac{\partial^2}{\partial r^2}$, $\frac{\partial^2}{\partial \theta^2}$, $\frac{\partial^2}{\partial \theta \partial r}$, $\frac{\partial}{\partial r}$, and $\frac{\partial}{\partial \theta}$. (Hint: You will need the Product Rule for differentiation here.)

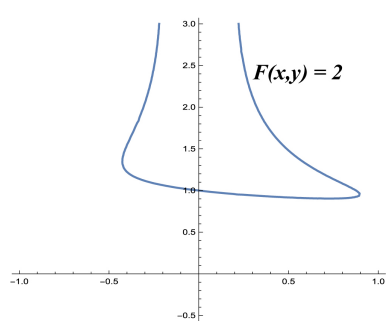

FIGURE 0.1. The 2-level set of $F(x, y) = \sin(xy) - x^2y^7 + 2e^{2y-2}$, in part (b) of Question 4.

Question 4. The equation

$$F(x, y) = 0,$$

defines y implicitly as a function of x, so we can write $y(x)$. Do the following:

(a) If both F and $y(x)$ are assumed to be C^1, show that

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}$$

provided that $F_y(x, y) \neq 0$.

(b) The 2-level set of $F(x, y) = \sin(xy) - x^2y^7 + 2e^{2y-2}$ is shown in the graph in Figure 1. Use **part (a)** to find the equation of the line tangent to this curve at the point $(0, 1)$.

Question 5. The equation

$$G(x, y, z) = 0,$$

defines z implicitly as a function of x and y, so we can write $z(x, y)$. Do the following:

(a) If both G and $z(x, y)$ are assumed to be C^1, show that

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)}, \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}$$

provided that $F_z(x, y, 0) \neq 0$.

(b) Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, where z is given implicitly by the equation

$$x^3z + y\cos z + \frac{\sin y}{z} = 0.$$

Question 6. Suppose $f(x, y)$ is a differentiable function of three variables.

(a) Explain what the quantity $\nabla f(x, y, z) \cdot (-\mathbf{k})$ represents.

(b) How does $\nabla f(x, y, z) \cdot (-\mathbf{k})$ relate to $\frac{\partial f}{\partial z}$?

Question 7. Calculate the directional derivative of $f(x, y, z) = \dfrac{xe^y}{3z^2 + 1}$ at the point $\mathbf{a} = (2, -1, 0)$ in the direction of $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

Question 8. Calculate the plane tangent to the surface whose equation is $x^2 - 2y^2 + 5xz = 7$ at the point $\left(-1, 0, -\frac{6}{5}\right)$ in two ways:

(a) By solving for z in terms of x and y and using the partial derivative of z, and

(b) By considering this surface the level set of a function and using the function's gradient.

Question 9. Do the following for the surface defined by $x^3z + x^2y^2 + \sin(yz) = -3$:

(a) Find an equation for the tangent plane to the surface at the point $(-1, 0, 3)$.

(b) Find a set of parametric equations defining the line *normal* to the surface at the point $(-1, 0, 3)$. (The line normal to a surface at a point is the line perpendicular to the tangent space of the surface at that point that passes through the point.

(c) Find a general formula for the normal line to a surface defined by $F(x, y, z) = 0$ at the point (x_0, y_0, z_0).

Question 10. Let $F(x, y) = c$ define a curve S_c in the plane. Suppose (x_0, y_0) is a point of S_c such that $\nabla F(x_0, y_0) \neq \mathbf{0}$. Show that near (x_0, y_0) the curve S_c can either be represented as the graph of a function $y = f(x)$ or of a function $x = g(y)$

Question 11. Let S be the points defined by the equation $\sin xy + e^{xz} + x^3 y = 1$. Near which points can we describe S as the graph of a C^1 function $z = f(x, y)$? Can you find an expression for $f(x, y)$ in this case? Now describe the set of points $(x_0, y_0, z_0) \in S$ where you cannot describe S as the graph of a function $z = f(x, y)$.