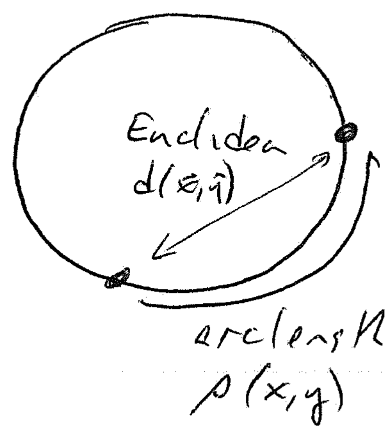
Lecture 4.

Examples of metrics ~~[illegible]~~ in $\mathbb{R}^n$ include:

- Euclidean: $d(\vec{x},\vec{y}) = \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}$
- Manhattan (taxicab): $d(\vec{x},\vec{y}) = \sum_{i=1}^{n}|x_i-y_i|$
- Maximum: $d(\vec{x},\vec{y}) = \max_i |x_i-y_i|$

ex. On a circle of radius r in $\mathbb{R}^2$, 2 metrics that are easy to "see":



Here, for x,y antipodal pts:

$$d(\vec{x},\vec{y}) = 2r$$
$$\rho(x,y) = \pi r$$

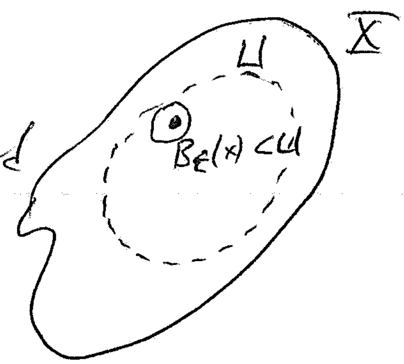
With a metric on X, one can define an open and closed ball in X:

open $\quad B_r(x) = \{ y \in X \mid d(x,y) < r \}$

closed $\quad \overline{B_r}(x) = \{ y \in X \mid d(x,y) \leq r \}$

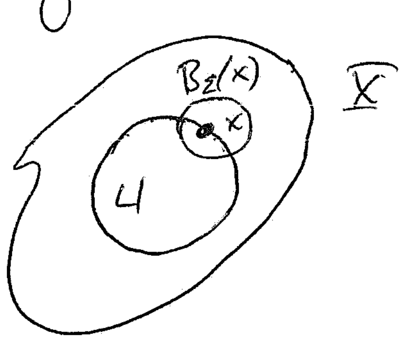
Note: We call these r-balls because in the Euclidean metric they look like n-dimensional balls, with spheres as skins. But in different metrics, balls (circles) will not look round! See the text exercises 29 and 30.

With the definition of an r-ball, we can define general open and closed sets, and neighborhoods:



Def. A subset $\mathcal{U} \subset \mathbb{X}$ is called <u>open</u> if $\forall x \in \mathcal{U}$, $\exists\, \varepsilon > 0 \ni B_\varepsilon(x) \subset \mathcal{U}$.

Def. A pt $x \in \mathbb{X}$ is a <u>boundary pt</u> of a subset $\mathcal{U} \subset \mathbb{X}$ if $\forall \varepsilon > 0$, $B_\varepsilon(x)$ contains at least one pt in $\mathcal{U}$ and at least one pt not in $\mathcal{U}$.



<u>Defn</u> A subset $U \subset X$ is called closed in X if every boundary pt of U is an element of U.

<u>Def</u> A subset $U \subset X$ is called a <u>neighborhood</u> of a point $x \in X$ if $\exists\, \varepsilon > 0$ where $B_\varepsilon(x) \subset U$.

<u>Def.</u> The set of all boundary pts of a subset $U \subset X$ is called the boundary of U and denoted ∂U.

<u>Final notes on metrics</u>

① A set w/ a metric on it is called a <u>metric space</u>.

② Any subset of a metric space is a metric space.

③ If $X \subset \mathbb{R}^n$, we can always assume that it is a metric space.

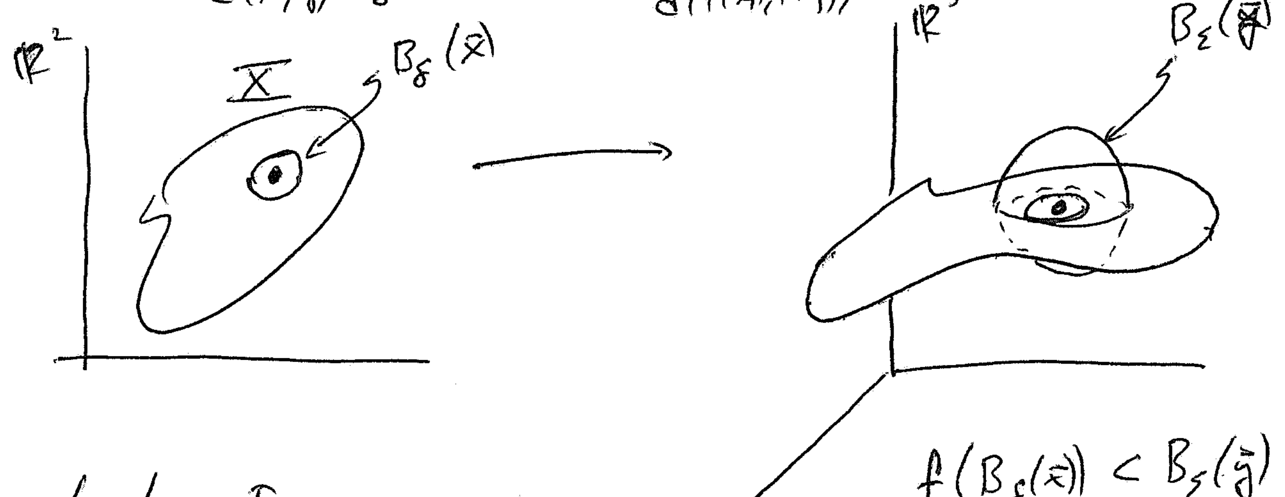
Recall the definition of (function) continuity from vector calculus:

$f : X \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $x \in X$ if

$\forall \varepsilon > 0, \exists \delta > 0 \ni$ if

$\| \vec{x} - \vec{y} \| < \delta$, then $\| f(\vec{x}) - f(\vec{y}) \| < \varepsilon$

$d(\vec{x}, \vec{y}) < \delta$ $\qquad$ $d(f(\vec{x}), f(\vec{y})) < \varepsilon$.



$f(B_\delta(\vec{x})) \subset B_\varepsilon(\vec{y})$.

In calculus I, $m = n = 1$

and the metrics was the same.

In vector calculus, the metrics in the above def are different: $\quad d(\vec{x}, \vec{y}) = d_{\mathbb{R}^n}(\vec{x}, \vec{y})$

$$d(f(\vec{x}), f(\vec{y})) = d_{\mathbb{R}^m}(f(\vec{x}), f(\vec{y}))$$

__New def__ A ~~mapping~~ function $f: X \subset \mathbb{R}^n \to \mathbb{R}^m$ is called __Lipschitz cont__, with constant λ, or λ-Lipschitz, if

$$d(f(x), f(y)) \leq \lambda\, d(x,y) \quad \forall x, y \in X$$

__Notes__ ① λ is a bound on the stretching ability of f on X.

② Lipschitz continuity is "stronger" than continuity. (If f is Lip cont. $\Rightarrow$ f is cont.)

③ Obvious? that $\lambda \geq 0$

④ If f is λ_0-Lipschitz, then f is λ-Lip for all $\lambda > \lambda_0$. So

define $\displaystyle \mathrm{Lip}(f) = \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x,y)}$

as the best one.

ex. $f(x) = \frac{1}{x}$ is Lipschitz cont on any domain (a,b), where $0 < a \le b \le \infty$, but not on $(0,b)$ for any $b > 0$.

ex. $h(x) = |x|$ is Lipschitz cont on $\mathbb{R}$.
What is $Lip(h)$?

ex. degree $n \geq 1$ polynomials are never Lip cont on infinite length intervals and always Lip cont on finite length ones.

Notice Let $f: \mathbb{R} \to \mathbb{R}$ be λ-Lipschitz. Then the condition is $|f(x) - f(y)| \le \lambda |x-y| \;\; \forall x,y \in \mathbb{R}$.

and then for $x \neq y$, $\dfrac{|f(x) - f(y)|}{|x-y|} \le \lambda$.

If f is diff. at x, then LHS limits to $|f'(x)|$ as $y \to x$.

$\Longrightarrow$ For diff. functions, the Lip. constant bounds the derivative! (even tho Lip cont mentions no der)

We use this

Prop Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be C^1 on an open I,
where $\forall x \in I$, $|f'(x)| \leq \lambda$.
$\implies f$ is λ-Lipschitz

pf. This is just an application of the MVT.
$$d(f(x), f(y)) = |f(x) - f(y)| \overset{MVT}{=} |f'(c)|\,|x-y|$$
for some $c \in (x, y)$, and
$$|f'(c)|\,|x-y| \leq \lambda |x-y| = \lambda\, d(x,y) \quad \blacksquare.$$

Def A λ-Lipschitz function $f: X \subset \mathbb{R}^n \to \mathbb{R}^m$ on a metric space X is called a contraction if $\lambda < 1$.

For our purposes, we will always assume codomain and domain are the same!

ex. The map $f: \mathbb{R} \to \mathbb{R}$, $f(x) = e^k x$, $k < 0$ (this is the time-1 map of $\dot{x} = kx$), is a contraction, with $\lambda = e^k < 1$.

ex. $g(x) = \sqrt{x}$ is a contraction on $(1, \infty)$. What is $Lip(g)$ here? $\lambda = \frac{1}{2}$.

<u>Prop</u> Let f be C^1 on a closed, bounded interval, with $|f'(x)| < 1$ $\forall x \in I$. $\Rightarrow$ f is a contraction.

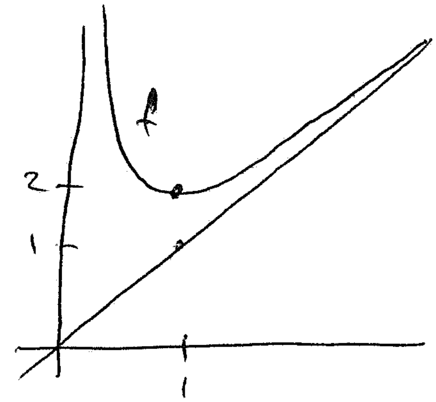
pf. f' is continuous on I, so by EVT will achieve its max. of $\lambda_m < 1$. By previous <u>prop</u> then f is λ_m-Lipschitz. ∎

Note: Closed and bounded are both vital for this to be true.

Closed, not bdd counter example

$f(x) = x + \frac{1}{x}$ on $I = [1,\infty)$

why not?



bdd but not closed counter example

$g(x) = 2\sqrt{x}$ on $(1,b)$ for any $b > 4$

(why must we need $b>4$ here?).

Lipschitz cont says alot about how fast the images of pts can move away from each other (exponential at best!!), since

$$\underbrace{d(f^2(x), f^2(y)) \le \lambda\, d(f(x), f(y)) \le \lambda(\lambda d(x,y)) = \lambda^2 d(x,y).}_{}$$

$$d(f^n(x), f^n(y)) \le \lambda^n d(x,y), \quad n \in \mathbb{N}.$$

And if $\lambda < 1$ ...? For a contraction??

This is tailor made for studying orbit behavior!

The Contraction Principle

Thm Let $I \subset \mathbb{R}$ be closed and $f: I \to I$ a contraction. Then f has a unique fixed pt x_0, and $\forall x \in I$,

$$|f^n(x) - x_0| \leq \lambda^n |x - x_0|$$

Notes ① Thus $\forall x \in I$, $\mathcal{O}_x \longrightarrow x_0$ exponentially due to λ^n, $0 < \lambda < 1$.

② For now, only valid in $\mathbb{R}$.

③ To show this, we will need the idea of a sequence in $\mathbb{R}$ ~~[illegible]~~ being Cauchy.

Def. For $N \in \mathbb{N}$, a sequence $\{x_i\}_{n=1}^{\infty}$ in $\mathbb{R}^N$ is called Cauchy if $\forall \varepsilon > 0$, $\exists A > 0 \ni \forall m, n \geq A$, $d(x_m, x_n) < \varepsilon$.