Homework

1. Consider the following four functions. Determine which extend (i.e., can be analytically continued) to an entire function (i.e., a holomorphic function on the entire complex plane) and which cannot. Remember to justify your answers.

   (a) (10 points) $f_1(x+iy) = u(x,y) + iv(x,y)$ where $u(x,y) = x^2 + y^2 - 1$ and $v(x,y) = 2xy$. Hint: Consider the Cauchy-Riemann equations.

   (b) (10 points) $f_2(z) = z^2 - 1$.

(c) (10 points) $f_3(z) = \sum_{n=1}^{\infty} n^{-n}(z-2)^n$

(d) (10 points) $f_4(z) = \sum_{n=1}^{\infty} n^2 z^n$.

2. (a) (20 points) Use a contour integral to carry out the the following computation:

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx = \frac{\pi}{e}.$$

Hint: Use as contours the semi-circles of radius R with keyhole at $z = i$ and that if $z = x + iy$ and $y \geq 0$, then $|\exp(iz)| = \exp(-y) \leq 1$.

3. (a) (10 points) Compute $\int_{\partial D_1(0)} \bar{z} dz$ where $\partial D_1(0)$ is the unit circle with positive orientation. Use this to explain why there is no holomorphic function $f : D_2(0) \to \mathbb{C}$ with $f(z) = \bar{z}$ for $z \in \partial D_1(0)$.

(b) (10 points) Show that if $g : D_2(0) \setminus \{0\} \to \mathbb{C}$ is holomorphic and $g(z) = \bar{z}$ for $z \in \partial D_1(0)$, then $g(z) = \frac{1}{z}$. Why does this not contradict part a)?

4. (20 points) Use the Cauchy inequalities to show that if f is an entire function that satisfies

$$|f(z)| \le C|z| \log(1 + |z|),$$

for all $z \in \mathbb{C}$, then $f(z) = 0$ for all $z \in \mathbb{C}$. Hint: Determine $\lim_{r\to\infty} \frac{\log(1+r)}{r}$ and $\lim_{r\to 0^+} \frac{\log(1+r)}{r}$ and use the first limit to show $f(z) = az + b$.