Exam 1 — 110.109 Calculus II — October 15, 2008

Instructions: You must show all of your work clearly and legibly to receive full credit. No calculators, books, or notes are allowed for this exam. You will find a formula sheet that you can use attached to this exam. Please print your name and section number.

NAME: SECTION:

I will not cheat.
Signature:

Problem 1:

Problem 2:

Problem 3:

Problem 4:

Problem 5:

Total:

1. *a*) : Evaluate the integral $\int \tan^3 x \sec x dx$.

$$\int \tan^3 x \sec x \, dx = \int \tan^2 x \tan x \sec x \, dx$$

$$= \int (\sec^2 x - 1) \, d(\sec x)$$

$u = \sec x$

$$= \int u^2 - 1 \, du$$

$$= \frac{u^3}{3} - u + C$$

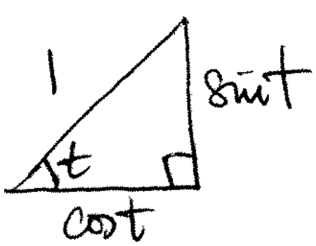
$$\therefore = \frac{\sec^3 x}{3} - \sec x + C.$$

b) : Evaluate the integral $\int \frac{1}{x^2\sqrt{x^2-4}} dx$.

$x = 2\sec t$

$$\int = \int \frac{2\sec t \tan t \, dt}{2^2 \sec^2 t \sqrt{4\sec^2 t - 4}} = \int \frac{\tan t \, dt}{4\sec t \cdot \tan t}$$

$$= \int \frac{1}{4\sec t} dt = \frac{1}{4}\int \cos t \, dt = \frac{1}{4}\sin t + C$$



$$\cos t = \frac{2}{x} \therefore \sin t = \sqrt{1 - \frac{4}{x^2}} = \frac{\sqrt{x^2-4}}{x}$$

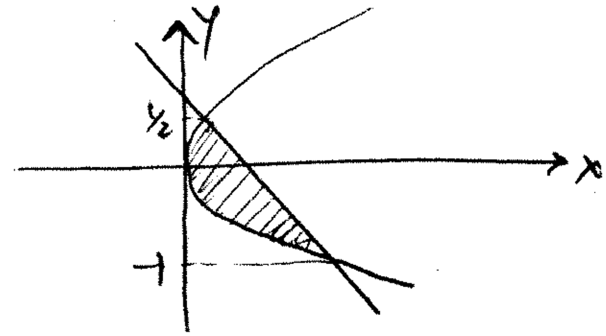
$$\therefore \int = \frac{\sqrt{x^2-4}}{4x} + C.$$

c) : Evaluate the integral $\int \frac{1}{x^2(x^2+1)}dx$.

$$\frac{1}{x^2(x^2+1)} = \frac{x^2+1-x^2}{x^2(x^2+1)} = \frac{1}{x^2} - \frac{1}{x^2+1}.$$

$$\int \frac{1}{x^2(x^2+1)}dx = \int \frac{dx}{x^2} - \int \frac{dx}{x^2+1} = -x^{-1} - \tan^{-1}x + C.$$

2. *a)* : Sketch the region enclosed by the curves $x = 2y^2, x + y = 1$. Find the area of the region.



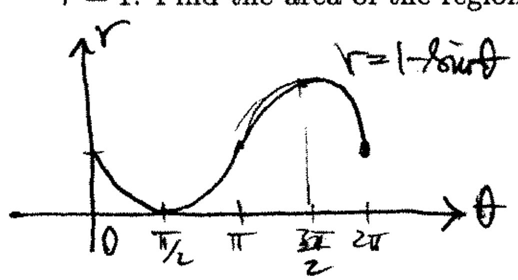
$2y^2+y=1 \quad \therefore 2y^2+y-1=0$

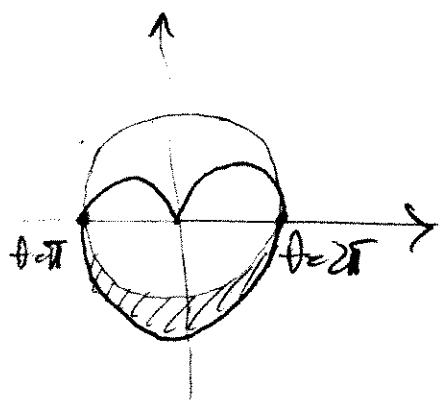
$\therefore y=-1,\ y=\frac{1}{2}.$

$$\int_{-1}^{1/2}(1-y)-2y^2\,dy = y-\frac{y^2}{2}-\frac{2y^3}{3}\Big|_{-1}^{\frac{1}{2}}$$

$$=\left(\frac{1}{2}-\frac{1}{8}-\frac{1}{12}\right)-\left(-1-\frac{1}{2}+\frac{2}{3}\right)=\frac{7}{24}+\frac{5}{6}=\frac{9}{8}.$$

b) : Sketch the region that lies inside the curve $r = 1 - \sin\theta$ and outside the curve $r = 1$. Find the area of the region.





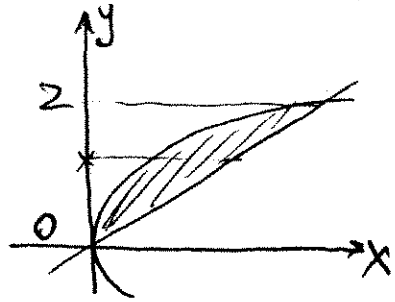
$$\int_{\pi}^{2\pi}\frac{1}{2}\left((1-\sin\theta)^2-1\right)d\theta =$$

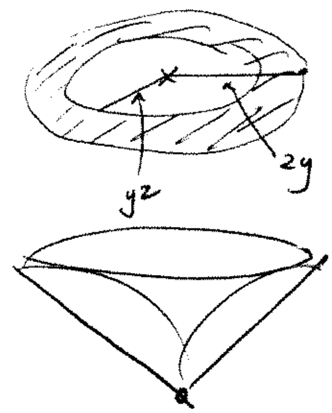
$$=\int_{\pi}^{2\pi}-\sin\theta+\frac{\sin^2\theta}{2}\,d\theta = \cos\theta\Big|_{\pi}^{2\pi}+\int_{\pi}^{2\pi}\frac{1-\cos 2\theta}{4}\,d\theta$$

$$=2+\left(\frac{\pi}{4}-\frac{\sin 2\theta}{8}\Big|_{\pi}^{2\pi}\right)=2+\frac{\pi}{4}.$$

3. *a)* : Find the volume of the solid obtained by rotating the region bounded by the curves $y^2 = x, x = 2y$ about the y-axis. Sketch the region, the solid, and a typical disk or washer.

$2y = y^2 : y = 0, 2$



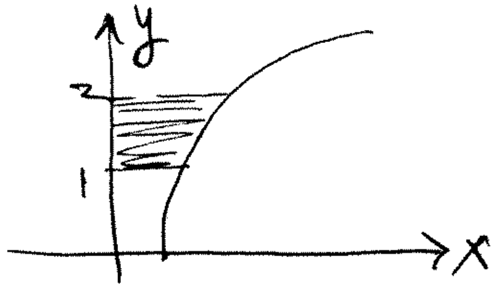


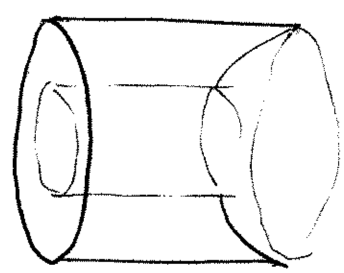
$$\int_0^2 \pi\left((2y)^2 - (y^2)^2\right) dy =$$

$$= \pi \int_0^2 4y^2 - y^4 \, dy = \pi\left(\frac{4y^3}{3}\Big|_0^2 - \frac{y^5}{5}\Big|_0^2\right)$$

$$= \pi\left(\frac{2^5}{3} - \frac{2^5}{5}\right) = \pi \cdot \frac{2^6}{15}.$$

b) : Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by the curves $x = 1 + y^2, x = 0, y = 1, y = 2$ about the x-axis. Sketch the region and a typical shell.



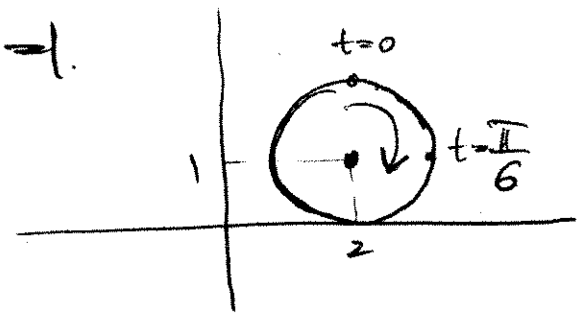


$$V = \int_1^2 2\pi y(1+y^2)\,dy = 2\pi \frac{y^2}{2}\Big|_1^2 + \frac{2\pi}{3} y^3\Big|_1^2$$

$$= 2\pi\left(2 - \frac{1}{2}\right) + \frac{2\pi}{3} \cdot 7 = 3\pi + \frac{14\pi}{3} = \frac{23\pi}{3}.$$

4. *a)* : Consider the parametric curve $x = 2 + \sin(3t), y = 1 + \cos(3t)$. Eliminate the parameter to find a Cartesian equation of the curve. Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

$$(x-2)^2+(y-1)^2 = \sin^2(3t)+\cos^2(3t) = 1.$$



b) : Consider the parametric curve $x = 4 + t^2, y = t^2 + t^3$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. For which values of t is the curve concave upward?

$$y=F(x) \quad t^2+t^3 = F(4+t^2). \quad \Big| \frac{d}{dt}:$$

$$2t+3t^2 = F'(4+t^2)\cdot(2t) \quad \therefore F'(4+t^2) = 1+\frac{3}{2}t.$$

$$\frac{d}{dt} \hookrightarrow 2+6t = F''(4+t^2)(4t^2)+F'(4+t^2)\cdot 2 = F''(4+t^2)(4t^2)+2+3t$$

$$\therefore F''(4+t^2) = \frac{3}{4t}.$$

$$\frac{dy}{dx} = 1+\frac{3}{2}t \quad \frac{d^2y}{dx^2} = \frac{3}{4t}.$$

C conc. up. when $t>0$.

5. *a)* : Find the length of the curve $y = \ln(\sec x), 0 \le x \le \frac{\pi}{4}$.

$$\frac{dy}{dx} = \frac{1}{\sec x} \cdot \sec x \tan x = \tan x$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2 x = \sec^2 x$$

$$L = \int_0^{\pi/4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{\pi/4} \sec x \, dx$$

$$= \ln|\sec x + \tan x| \Big|_0^{\pi/4}$$

$$= \ln(1+\sqrt{2}) - \ln(1)$$

$$= \ln(1+\sqrt{2}).$$

b) : Find the length of the parametric curve $x = 1 + 3t^2, y = 4 + 2t^3, 0 \le t \le 1$.

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\, dt$$

$$= \int_0^1 6t\sqrt{1+t^2}\, dt$$

$$u = 1 + t^2 \quad du = 2t\, dt$$

$$L = \int_1^2 3\sqrt{u}\, du = 3 \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} \Big|_1^2 = 2\left(2^{\frac{3}{2}} - 1\right).$$

$c)$: Sketch the region of the points whose polar coordinates satisfy $|r| \leq 1, |\theta| \leq \frac{\pi}{4}$.