Exam 2 — 110.109 Calculus II — November 24, 2008

Instructions: You must show all of your work clearly and legibly to receive full credit. No calculators, books, or notes are allowed for this exam. You will find a formula sheet that you can use attached to this exam. Please print your name and section number.

NAME: SECTION:

I will not cheat.
Signature:

Please do not write on this table!

Problem	Points	Score
1	40	
2	30	
3	20	
4	20	
5	20	
TOTAL	130	

1. Find the limit of the following sequences. If the limit does not exist, explain why.

$a) : \{1 - (0.99)^n\}_{n\geq 1}$.

$$|0.99| < 1 \Rightarrow \lim_{n\to\infty} (0.99)^n = 0.$$

$$\Rightarrow 1 - (0.99)^n \xrightarrow[n\to\infty]{} 1 - 0 = 1.$$

$b) : \{\frac{\sin(n)}{n}\}_{n\geq 1}$.

$$\left|\frac{\sin n}{n}\right| \leq \frac{1}{n} \quad : \quad -\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$$

$$\lim_{n\to\infty}\left(-\frac{1}{n}\right) = \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0.$$

By the squeeze theorem, $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$.

$c) : \{(1-\frac{1}{n})^n\}_{n\geq 1}.$

$$\log\left(\left(1-\frac{1}{n}\right)^n\right) = n\log\left(1-\frac{1}{n}\right) = \frac{\log(1-\frac{1}{n})}{\frac{1}{n}}$$

$$\lim_{x\downarrow 0}\frac{\log(1-x)}{x} \underset{\uparrow}{=} \lim_{x\downarrow 0}\frac{\frac{1}{1-x}}{1} = -1$$

L'Hospital $\frac{0}{0}$

$$\Rightarrow \lim_{n\to\infty}\log\left(\left(1-\frac{1}{m}\right)^n\right) = -1$$

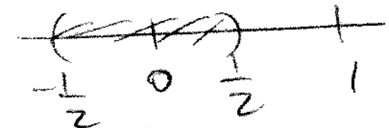
$$\Rightarrow \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \frac{1}{e}.$$

$d) : \{0, 1, 0, 1, 0, 1, 0, 1, 0, 1, \ldots\}.$

$$a_n = \begin{cases} 0 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$$

Suppose limit exists. $a_n \to a$. Then $a=0$ or $a=1$.

Say $a=0$.



Then $a_n \in (-\frac{1}{2}, \frac{1}{2})$ for $n \geq n_0$. Choose even $n \geq n_0$

$\Rightarrow 1 \in (-\frac{1}{2}, \frac{1}{2})$. Contradiction.

Same argument for $a=1$.

$\Rightarrow$ limit does not exist.

2. a) Write $0.1\overline{3}$ as a ratio of two positive integers.

$$0.1333\ldots = \frac{1}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots$$

$$= \frac{1}{10} + \frac{3}{10^2}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \cdots\right)$$

$$= \frac{1}{10} + \frac{3}{10^2} \cdot \frac{1}{1 - \frac{1}{10}}$$

$$= \frac{12}{90}$$

$$= \frac{2}{15}$$

b) A series $\sum_{n=1}^{\infty} a_n$ has partial sums $s_n = \frac{n-1}{n+1}$. Find a_n and $\sum_{n=1}^{\infty} a_n$.

$a_1 = s_1 = 0.$

$a_n = s_n - s_{n-1} \quad (n \geq 2)$

$$= \frac{n-1}{n+1} - \frac{n-2}{n} = \frac{2}{n(n+1)}.$$

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{n-1}{n+1} = 1.$$

3. Decide convergence/divergence. **Explain!** (you are not required to show absolute convergence/divergence).

a) $\sum_{n=1}^{\infty} \cos(\frac{1}{n})$.

$$\lim_{x\to 0} \cos x = 1 \Rightarrow \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = 1$$

No Name Test $\Rightarrow \sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ divergent.

b) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1} = -\sum_{n=1}^{\infty} (-1)^{n-1} b_n, \quad b_n = \frac{n}{n^2+1} = \frac{1}{n+\frac{1}{n}}$$

- $b_n \geq 0$
- $b_n \downarrow$: same as $1/b_n \uparrow$

$$n + \frac{1}{n} \underset{?}{\leq} n+1+\frac{1}{n+1}$$

$$\frac{1}{n} \underset{?}{\leq} 1 + \frac{1}{n+1} \quad \text{YES}$$

- $b_n \to 0$? : $b_n = \frac{1}{n+\frac{1}{n}} \xrightarrow[n\to\infty]{} 0$ ✓.

Alternating Series Test $\Rightarrow \sum (-1)^{n-1} b_n$ convergent

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ convergent.

c) Compute $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$.

$$\frac{2}{n^2-1} = \frac{1}{n-1} - \frac{1}{n+1}$$

$$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$

$$\left(1-\cancel{\tfrac{1}{3}}\right)+\left(\tfrac{1}{2}-\cancel{\tfrac{1}{4}}\right)+\left(\cancel{\tfrac{1}{3}}-\cancel{\tfrac{1}{5}}\right)+\left(\cancel{\tfrac{1}{4}}-\cancel{\tfrac{1}{6}}\right)+\left(\cancel{\tfrac{1}{5}}-\tfrac{1}{7}\right)+\left(\cancel{\tfrac{1}{6}}-\cancel{\tfrac{1}{8}}\right)+\cdots$$

$$\xrightarrow[n\to\infty]{} 1+\frac{1}{2} = \frac{3}{2}.$$

4. a) Show that $\sum_{n=1}^{\infty} ne^{-n}$ converges.

$f : [1,\infty) \to \mathbb{R}$, $f(x) = xe^{-x}$

- f cts
- $f \geq 0$
- $f \downarrow$? ; $f'(x) = (1-x)e^{-x} \leq 0$ ✓.

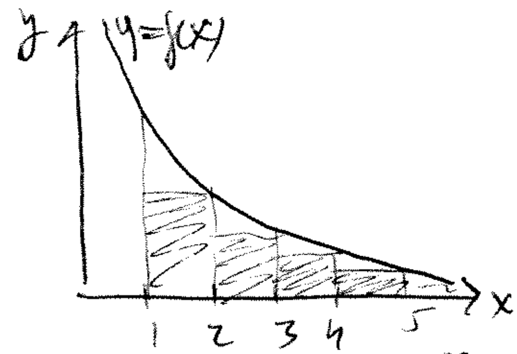
$$\int_1^t xe^{-x}dx = \int_1^t x\,d(-e^{-x}) = -xe^{-x}\Big|_1^t - \int_1^t -e^{-x}dx$$

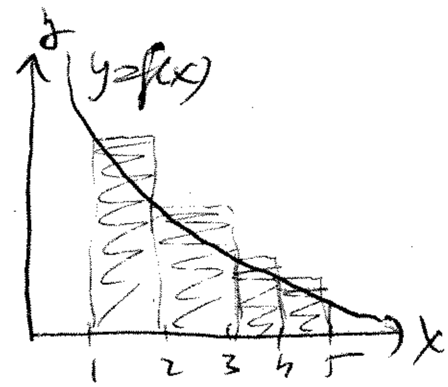
$$= -(1+x)e^{-x}\Big|_1^t = \frac{2}{e} - \frac{1+t}{e^t}$$

$$\int_1^\infty xe^{-x}dx = \lim_{t\to\infty}\int_1^t xe^{-x}dx = \frac{2}{e}$$

Integral Test $\Rightarrow \sum_{n=1}^{\infty} ne^{-n}$ converges.

b) Show that $\frac{2}{e} \leq \sum_{n=1}^{\infty} ne^{-n} \leq \frac{3}{e}$.





$$\sum - (1^{st}\text{ term}) \leq \int_1^\infty = \frac{2}{e}$$

$$\Downarrow$$

$$\sum \leq \frac{3}{e}$$

$$\int_1^\infty \leq \sum$$

$$\Downarrow$$

$$\frac{2}{e} \leq \sum$$

5. a) Compute $\int_1^\infty \frac{\ln x}{x^2} dx$

$$\int_1^t \frac{\ln x}{x^2} dx = \int_1^t \ln x \, d\left(-\frac{1}{x}\right) = -\frac{\ln x}{x}\Big|_1^t + \int_1^t \frac{1}{x^2} dx$$

$$= -\frac{1+\ln x}{x}\Big|_1^t$$

$$= 1 - \frac{1+\ln t}{t}$$

$$\int_1^\infty \frac{\ln x}{x^2} dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^2} dx = 1 - 0 = 1.$$

b) Decide convergence/divergence for $\int_1^\infty \frac{x}{\sqrt{1+x^6}} dx$

$$0 \le \frac{x}{\sqrt{1+x^6}} \le \frac{x}{\sqrt{x^6}} = \frac{x}{x^3} = \frac{1}{x^2}.$$

$\int_1^\infty \frac{1}{x^2} dx$ converges.

Comparison Test $\Rightarrow \int_1^\infty \frac{x}{\sqrt{1+x^6}} dx$ converges.