110.107 Calculus 2 - Mese
Exam 1, March 5, 2008

Directions. You must show all of your work clearly and legibly to receive full credit. No calculators, books, or notes are allowed for this exam. You will find a formula sheet that you can use attached to this exam. Please print your name and section number.

NAME: SECTION:

I will not cheat.
Signature:

Solution set

Problem 1:

Problem 2:

Problem 3:

Problem 4:

Total:

1. (a) Find the partial fraction decomposition of $\frac{5x^3-3x^2+7x-3}{(x^2+1)^2}$.

$$\frac{5x^3-3x^2+7x-3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} = \frac{(Ax+B)(x^2+1)+Cx+D}{(x^2+1)^2}$$

$$= \frac{Ax^3+Bx^2+(A+C)x+(B+D)}{(x^2+1)^2}$$

$5 = A$

$-3 = B$

$7 = A+C \Rightarrow C=2$

$-3 = B+D \Rightarrow D=0.$

$$\therefore \frac{5x^3-3x^2+7x-3}{(x^2+1)^2} = \frac{5x-3}{(x^2+1)} + \frac{2}{(x^2+1)^2}$$

(b) Evaluate the integral $\int \frac{2x+1}{x^2+4x+8}\,dx$

Let $u = x^2+4x+8$

$du = 2x+4$

$$\int \frac{2x+1}{x^2+4x+8}\,dx = \int \frac{2x+4}{x^2+4x+8}\,dx - \int \frac{3}{x^2+4x+8}\,dx$$

$$= \int \frac{du}{u} - 3\int \frac{1}{(x^2+4x+4)+4}\,dx$$

$$= \ln|u| - 3\int \frac{1}{(x+2)^2+4}\,dx$$

$$= \ln|x^2+4x+8| - 3\arctan\frac{x+2}{2} + C.$$

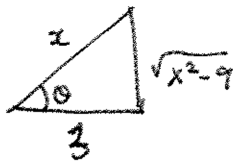
(c) Evaluate the integral $\int \frac{\sqrt{x^2-9}}{x}\,dx$.

$x = 3\sec\theta \qquad \sqrt{x^2-9} = \sqrt{9\sec^2\theta - 9} = 3\tan\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$

$$\int \frac{\sqrt{x^2-9}}{x}\,dx = \int \frac{3\tan\theta}{3\sec\theta}\cdot 3\sec\theta\tan\theta\,d\theta$$

$$= 3\int \tan^2\theta\,d\theta = 3\int(\sec^2\theta - 1)\,d\theta$$



$$= 3\tan\theta - 3\theta + C$$

$$= 3\frac{\sqrt{x^2-9}}{3} - 3\sec^{-1}\left(\frac{x}{3}\right) + C.$$

(d) Evaluate the integral $\int \tan^3 x\ \sec^5 x\ dx$.

$$\int \tan^3 x \sec^5 x\,dx = \int \tan^2 x \sec^4 x\ (\sec x \tan x)\,dx$$

$$= \int(\sec^2 x - 1)\sec^4 x\ (\sec x \tan x)\,dx$$

$u = \sec x$

$du = \sec x \tan x\,dx$

$$= \int(\sec^6 x - \sec^4 x)\ (\sec x \tan x)\,dx$$

$$= \int(u^6 - u^4)\,du = \frac{u^7}{7} - \frac{u^5}{5} + C$$

$$= \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C.$$

(e) Evaluate the integral $\int \frac{x^2}{e^{2x}}\,dx$.

$dv = e^{-2x}dx \qquad u = x$

$v = -\frac{1}{2}e^{-2x} \qquad du = dx$

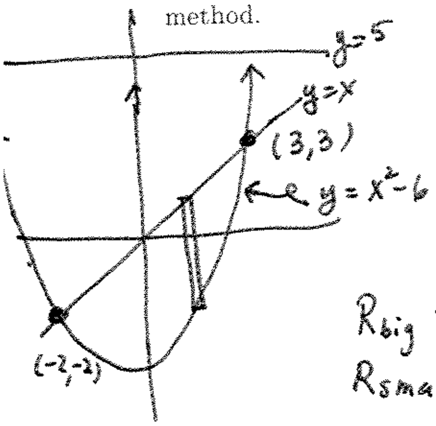
$\boxed{\begin{array}{ll} dv = e^{-2x}dx & u = x^2 \\ v = -\frac{1}{2}e^{-2x} & du = 2x \end{array}}$

$$\int \frac{x^2}{e^{2x}}dx = \int x^2 e^{-2x}dx = -\frac{x^2}{2}e^{-2x} + \int x e^{-2x}dx = -\frac{x^2}{2}e^{-2x} - \frac{x}{2}e^{-2x} + \frac{1}{2}\int e^{-2x}dx$$

$$= -\frac{x^2}{2}e^{-2x} - \frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + C.$$

2. (a) Let R be the region bounded by the curves $x^2 - y - 6 = 0$ and $y = x$. Write down (**but do not evaluate**) the the integral (or the sum of integrals) which gives the following:

(i) the volume of the solid obtained by rotating R about the line $y = 5$ using the disk or the washer method.



$x^2 - y - 6 = 0$ and $y = x \Rightarrow x^2 - x - 6 = 0 \Rightarrow (x-3)(x+2)=0$

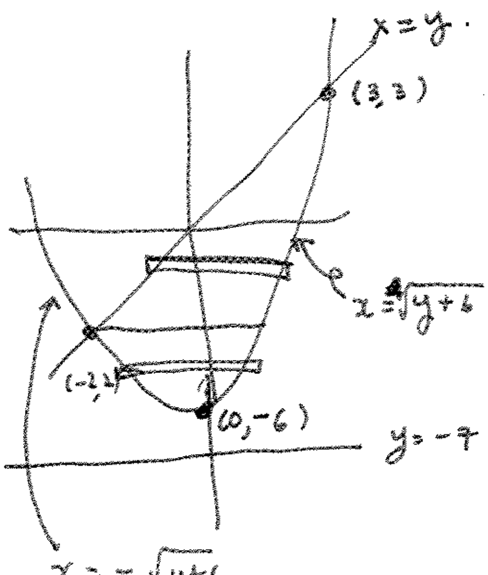
$\Rightarrow x = 3,\ x = -2$

$R_{big} = 5 - (x^2 - 6)$

$R_{small} = 5 - x$

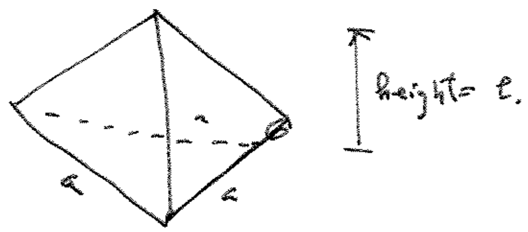
$$\pi \int_{-2}^{3} \left[(5-(x^2-6))^2 - (5-x)^2 \right] dx .$$

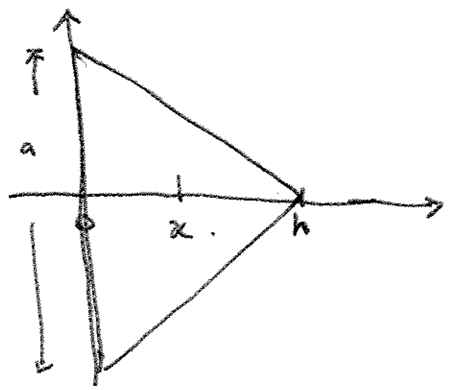
(ii) the volume of the solid obtained by rotating R about the line $y = -7$ using the cylindrical shell method.



$$\int_{-6}^{-2} \left[\sqrt{y+6} - (-\sqrt{y+6}) \right] dy + \int_{-2}^{3} \left[\sqrt{y+6} - y \right] dy$$

(b) Find the volume of a pyramid with height h and a base equilateral triangle with side length a.





The length of the sides of the equilateral triangle when slicing with a plane ~~of h~~ at height x is $\frac{a}{h}(h-x)$

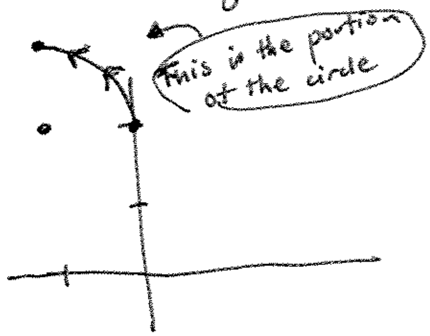
Area of a equilateral triangle with side length equal to $\frac{a}{h}(h-x)$ is $\frac{\sqrt{3}}{4}\frac{a}{h}(h-x)^2$. by elementary geometry.

$$\Rightarrow \int_0^h \frac{\sqrt{3}}{4}\frac{a}{h}(h-x)^2\,dx.$$

3. (a) Eliminate the parameter to find a Cartesian equation of the curve $x = \sin 2t - 1$, $y = \cos 2t + 2$ for $0 \le t \le \frac{\pi}{8}$. Sketch the curve given by the parametric equation.

$$\left.\begin{array}{l} x+1 = \sin 2t \\ y-2 = \cos 2t \end{array}\right\} \Rightarrow 1 = \sin^2 2t + \cos^2 2t = (x+1)^2 + (y-2)^2$$

$(x+1)^2 + (y-2)^2 = 1$ is the eqn of a circle of radius 1 centered at $(-1, 2)$.



For $0 \le t \le \frac{\pi}{8}$,
$0 \le x+1 \le 1$
$0 \le y+2 \le 1$

(b) Write down (**but do not evaluate**) the integral which gives the length of the portion of the curve $x = 3t - t^2$, $y = -t^2 + 6t - 8$ which lies in the first quadrant.

First quadrant $\Rightarrow$ $x \ge 0$, $y \ge 0$

$$3t - t^2 \ge 0 \qquad -t^2 + 6t - 8 \ge 0$$
$$t(3-t) \ge 0 \qquad -(t-4)(t-2) \ge 0$$
$$0 \le t \le 3 \qquad 2 \le t \le 4$$

$\therefore$ For both $x \ge 0$, $y \ge 0$, we must have $2 \le t \le 3$.

$$\text{Length} = \int_2^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_2^3 \sqrt{(3-2t)^2 + (-2t+6)^2}\, dt,$$

(c) Find $\frac{d^2y}{dx^2}$ when $x = \sqrt[3]{t}$ and $y = \sqrt[3]{t} - t$. $= t^{1/3} - t$

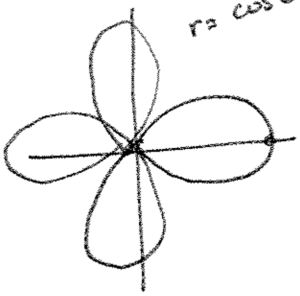
$$\frac{dy}{dt} = \frac{1}{3}t^{-2/3} - 1$$
$$\frac{dx}{dt} = \frac{1}{3}t^{-2/3}$$

$$\frac{dy}{dx} = \frac{\frac{1}{3}t^{-2/3} - 1}{\frac{1}{3}t^{-2/3}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\dfrac{\frac{-2}{9}t^{-5/3} \cdot \frac{1}{3}t^{-2/3} - \left(\frac{1}{3}t^{-2/3} - 1\right)\left(\frac{-2}{9}t^{-5/3}\right)}{\frac{1}{9}t^{-4/3}}}{\frac{1}{3}t^{-2/3}}$$

$$= \frac{\frac{-2}{27}t^{-7/3} - \left(\frac{1}{3}t^{-2/3} - 1\right)\left(\frac{-2}{9}t^{-5/3}\right)}{\frac{1}{27}t^{-2}}.$$

4. (a) Find the integral or the sum of integrals (**but do not evaluate**) which represents the area of the region bounded by one loop of the four leaved rose $r = \cos 2\theta$. Sketch the region.

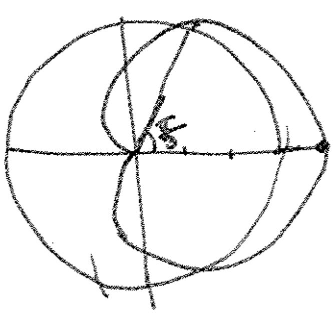


$$2\cdot\frac{1}{2}\int_0^{\pi/4} \cos^2 2\theta \, d\theta .$$

(b) Find the integral or the sum of integrals (**but do not evaluate**) which represents the area of the region which is inside both $r = 2 + 2\cos\theta$ and $r = 3$. Sketch the region.

Find intersection :

$2 + 2\cos\theta = 3 \Rightarrow 2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$.



$$2\cdot\frac{1}{2}\int_0^{\frac{\pi}{3}} (2+2\cos\theta)^2 - 9 \quad d\theta .$$

(c) From the formula $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, derive the formula $\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$ for the polar curve $r = f(\theta)$.

Use the identity $x = r\cos\theta$

$y = r\sin\theta$

product rule

$$\Rightarrow \begin{cases} \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \\ \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta \end{cases}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$