1. Determine whether or not the sequence $\{a_n\}$ converges, and find the limit if it does converge.

(a) $a_n = \frac{8n-7}{7n-8}$.

converges to $\frac{8}{7}$

(b) $a_n = 10 - (0.99)^n$.

converges to $10$

(c) $a_n = \frac{(\ln n)^3}{n^2}$.

converges to $0$

(d) $a_n = \frac{n-e^n}{n+e^n}$.

converges to $-1$.

2. Determine whether each infinite series converges or diverges. Clearly state which test you are using and show all work.

(a) $\sum_{n=0}^{\infty} \frac{3^n}{2^n + 4^n}$.

Use limit comparison with $\sum \frac{3^n}{4^n} = \sum \left(\frac{3}{4}\right)^n$.

$$\lim_{n\to\infty} \frac{\frac{3^n}{2^n+4^n}}{\frac{3^n}{4^n}} = \lim_{n\to\infty} \frac{4^n}{2^n+4^n} = \lim_{n\to\infty} \frac{1}{\frac{2^n}{4^n}+1} = 1$$

Hence either $\sum \frac{3^n}{2^n+4^n}$ and $\sum \frac{3^n}{4^n}$ both converge or both diverge.

Since $\sum \frac{3^n}{4^n}$ is a geometric series with $r=\frac{3}{4}$, $\sum \frac{3^n}{4^n}$ converge.

Therefore, $\sum \frac{3^n}{2^n+4^n}$ also converge.

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n^2}$

Use alternating series test.

Since $\frac{d}{dx}\left(\frac{\ln x}{x^2}\right) = \frac{\frac{1}{x}x^2 - (\ln x)(2x)}{x^4} = \frac{1-2(\ln x)}{x^3} < 0$ for $x \geq 2$,

$a_n = \frac{\ln n}{n^2}$ is decreasing for $n \geq 2$.

Futhermore, $\lim_{n\to\infty} \frac{\ln n}{n^2} = \lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{\frac{1}{x}}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$,

we can apply the alternating series test to conclude

$\sum (-1)^n \frac{\ln n}{n^2}$ converges.

(c) $\sum_{n=1}^{\infty} \frac{\sqrt{n} + \sqrt[3]{n}}{n^2 + n^3}$

Use limit comparison with $\sum \frac{n^{1/2}}{n^3} = \sum n^{-5/2}$

which converge because it is a $p$-series with $p = 5/2 > 1$.

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}+\sqrt[3]{n}}{n^2+n^3}}{\frac{1}{n^{5/2}}} = \lim_{n\to\infty} \frac{n^3 + n^{\frac{17}{6}}}{n^2+n^3} = \lim_{n\to\infty} \frac{1 + \frac{1}{n^{1/6}}}{\frac{1}{n}+1} = 1.$$

By the limit comparison test, either $\sum \frac{\sqrt{n}+\sqrt[3]{n}}{n^2+n^3}$ and $\sum \frac{1}{n^{5/2}}$ both converge or both diverge. ~~Thus,~~ Since $\sum \frac{1}{n^{5/2}}$ converge, $\sum \frac{\sqrt{n}+\sqrt[3]{n}}{n^2+n^3}$ also converge

(d) $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)}$

$$\int \frac{1}{x(\ln x)}\,dx = \int \frac{1}{u}\,du = \ln u + C = \ln(\ln x) + C$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$\int_3^{\infty} \frac{1}{x(\ln x)}\,dx = \lim_{t\to\infty} \int_3^t \frac{1}{x(\ln x)}\,dx = \lim_{t\to\infty} \left( \ln(\ln t) - \ln(\ln 3) \right) = \infty$$

$\Rightarrow \sum \frac{1}{n(\ln n)}$ divergent by the integral test

(e) $\sum_{n=0}^{\infty} \frac{n!}{e^n}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!} \right| = \lim_{n\to\infty} \frac{n+1}{e} = \infty$$

$\Rightarrow \sum \frac{n!}{e^n}$ divergent by the ratio test

3. Find the interval of convergence of each of the power series. Clearly state which test you are using and show all work.

(a) $\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$

Use ratio test to obtain the radius of convergence.

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{2^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n x^n} \right| = \lim_{n\to\infty} \frac{2}{n+1} |x| = 0 < 1.$

$\Rightarrow$ the radius convergence $= \infty$ (In other words, the ~~sequence~~ series converges for any choice of $x$).

$\Rightarrow$ the interval of convergence is $(-\infty, \infty)$.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{4n^2-1}$

Use ratio test to obtain the radius of convergence.

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{4(n+1)^2-1} \cdot \frac{4n^2-1}{x^n} \right| = \lim_{n\to\infty} \frac{4n^2-1}{4n^2+8n+3} \cdot |x|.$

$= |x|$

$\Rightarrow \begin{cases} \text{Converges for } |x|<1 \\ \text{diverges for } |x|>1 \end{cases}$ $\Rightarrow$ radius of convergence is 1.

Check endpoints

$\underline{x=1}$ : $\sum \frac{(-1)^n (1)^n}{4n^2-1} = \sum \frac{(-1)^n}{4n^2-1}$ converges by the alternating series test.

$\underline{x=-1}$ $\sum \frac{(-1)^n(-1)^n}{4n^2-1} = \sum \frac{1}{4n^2-1}$ converges by the comparison test w/ $\sum \frac{1}{4n^2}$. Since $\sum \frac{1}{n^2}$ converges (p-series with $p=2$), $\sum \frac{1}{4n^2-1}$ also converges. The interval of convergence is $[-1, 1]$.

(c) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n(3^n)}$

$$\lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n(3^n)}{(x-1)^n} \right| = \lim_{n\to\infty} \frac{n}{3(n+1)} \cdot |x-1| = \frac{1}{3}|x-1|$$

Convergent $\iff \frac{1}{3}|x-1| < 1 \iff |x-1| < 3 \iff -2 < x < 4$

Divergent $\iff \frac{1}{3}|x-1| > 1 \iff |x-1| > 3 \iff x > 4$ or $x < -2$.

Check endpoints

$x = 4$: $\sum_{n=1}^{\infty} \frac{3^n}{n(3^n)} = \sum_{n=1}^{\infty} \frac{1}{n}$ divergent (harmonic series or p-series with $p = 1$).

$x = -2$ $\sum \frac{(-3)^n}{n(3^n)} = \sum \frac{(-1)^n}{n}$ convergent by the alternating series test

Interval of convergence $[-2, 4)$.

4. (a) Use the fact that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$ to find a power series representation for $\ln(1-x)$. Explain your answer thoroughly.

See example 6, Sec 8.6

page 457 in text.

(b) How many terms of the series $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$ do we need to add in order to find the sum to an accuracy of $|\text{error}| < 0.3$. Explain your answer thoroughly.

Alternating series estimation theorem says $|R_n| \leq b_{n+1}$.

$$b_0 = 1$$
$$b_1 = \frac{2}{1!} = 2$$
$$b_2 = \frac{2^2}{2!} = 2$$
$$b_3 = \frac{2^3}{3!} = \frac{2 \cdot \cancel{2} \cdot 2}{3 \cdot \cancel{2}} = \frac{4}{3}$$
$$b_4 = \frac{2^4}{4!} = \frac{2 \cdot \cancel{2} \cdot \cancel{2} \cdot \cancel{2}}{\cancel{4} \cdot 3 \cdot \cancel{2}} = \frac{2}{3}$$
$$b_5 = \frac{2^5}{5!} = \frac{2 \cdot 2 \cdot \cancel{2} \cdot \cancel{2} \cdot \cancel{2}}{5 \cdot \cancel{4} \cdot 3 \cdot \cancel{2}} = \frac{4}{15} \approx 0.266\ldots < 0.3.$$

$.266$
$15\overline{)4}$
$30$
$100$
$90$

$\therefore$ You need to take $b_0 - b_1 + b_2 - b_3 + b_4$ to have an error of $|R_4| \leq b_5 = 0.266\ldots < 0.3$.

(c) Let $\{F_n\}$ be the Fibonacci sequence defined as $F_1 = 1$, $F_2 = 1$ and $F_{n+1} = F_{n-1} + F_n$. Show that $F_n \leq 2^n$ for all $n$.

Initial Step $\quad F_1 = 1 \leq 2^1$.

Inductive step

Assume $F_1 \leq 2^1, F_2 \leq 2^2, \ldots, F_{k-1} \leq 2^{k-1}, F_k \leq 2^k$.

Then $F_{k+1} = F_{k-1} + F_k \leq 2^{k-1} + 2^k \leq 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$.

$\square$