THE JOHNS HOPKINS UNIVERSITY
Faculty of Arts and Sciences
MIDTERM EXAM - SPRING SESSION 2009
110.109 - CALCULUS II.

Examiner: Professor C. Consani
Duration: 50 MINUTES (10am-10:50am), March 6, 2009.

No calculators, books, notes allowed. **Total Points = 100**

Student Name: ______________________________

Ethic Stat.: I agree to complete this exam without unauthorized assistance from any person, materials or device.

Student Signature: ___________________________

TA Name (circle one): A. Banerjee, S. Khan, A. Saltz

1.	
2.	
3.	
4.	
Total	

1.[25 points]

Find an antiderivative for the function

$$f(x) = (1 + x^2)^{-\frac{3}{2}}$$

Use the substitution $x = \tan(u)$, so that $dx = \sec^2(u)du$

$$\int \frac{dx}{(1+x^2)^{\frac{3}{2}}} = \int \frac{\sec^2(u)du}{(1+\tan^2(u))^{\frac{3}{2}}} = \int \frac{\sec^2(u)du}{(\sec^2(u))^{\frac{3}{2}}} = \int \frac{\sec^2(u)du}{(\sec^3(u))} = \int \frac{du}{sec(u)}$$

$$= \int cos(u)du = sin(u) + C$$

Using the similarity between two right triangles with a common angle u, we determine that

$$\sin(u) = \frac{x}{\sqrt{x^2+1}}$$

Therefore

$$\int \frac{dx}{(1+x^2)^{\frac{3}{2}}} = \frac{x}{\sqrt{x^2+1}} + C$$

2.[25 points] Let $\mathcal{R}$ be the region bounded by the curves $y = x^3$, $x = 0$ and $y = 1$.

a) Use the washer method to find the volume of the solid obtained by rotating $\mathcal{R}$ about the x-axis.

<u>**Solution**</u>: The outer edge of the washer is created by the line $y = 1$, so the outer radius is just 1. The inner radius is created by the curve $y = x^3$. Because we are rotating about the x-axis, we can see from a diagram that the inner radius is y and that the width of a washer is dx. The curves $y = x^3$ and $y = 1$ intersect at $(1, 1)$. Combining all this

$$V = \int_0^1 \pi(1)^2 - \pi(y)^2 dx = \int_0^1 \pi - \pi(x^3)^2 dx = \pi\left[x - \frac{x^7}{7}\right]_0^1 = \pi\left(1 - \frac{1}{7}\right) = \frac{6\pi}{7}$$

b) Find, by applying the method of cylindrical shells, the volume of the solid obtained by rotating the region $\mathcal{S}$ bounded by the curves $y = x^2 + 4$, $y = 2x^2$, and $x = 0$, about the $y-$axis.

<u>**Solution**</u>: We use cylindrical shells

$$\text{Volume} = 2\pi\int_0^2 \mathrm{x}(\mathrm{x}^2 + 4 - 2\mathrm{x}^2)\,\mathrm{dx} = 2\pi\int_0^2 4\mathrm{x} - \mathrm{x}^3\,\mathrm{dx} = 2\pi\left[2\mathrm{x}^2 - \frac{\mathrm{x}^4}{4}\right]_0^2 = 8\pi$$

3.[25 points]

Consider the curve given in parametric equations:

$$x(t) = 12t - t^3; \qquad y(t) = 6t^2$$

a) Determine the points on the curve at which the tangent lines are vertical and horizontal.

b) Find the area of the region contained inside the loop of the curve

Solution:

$$\frac{dy}{dx} = \frac{12t}{12 - 3t^2} = \frac{4t}{4 - t^2}$$

Hence: at $(x(0), y(0)) = (0, 0)$ the tangent line is horizontal and at $(x(\pm 2), y(\pm 2))$ the tangent lines are vertical

$$Area = 2\int_0^{2\sqrt{3}} (12t - t^3)(12t)dt = 24\left[4t^3 - \frac{1}{5}t^5\right]_0^{2\sqrt{3}} = \frac{2^9 3^2\sqrt{3}}{5}$$

4. [25 points]

Find the length of the arc of the curve

$$y = \ln(1 - x^2)$$

from the point $(0, 0)$ to the point $(\frac{1}{2}, \ln(3) - \ln(4))$

Solution: This is an application of the arc length formula

$$\frac{dy}{dx} = \frac{d}{dx}(\ln(1 - x^2)) = \frac{-2x}{1 - x^2}$$

It follows that the length of the arc is

$$L = \int_0^{\frac{1}{2}} \sqrt{(\frac{dy}{dx})^2 + 1}\ dx = \int_0^{\frac{1}{2}} \sqrt{(\frac{-2x}{1 - x^2})^2 + 1}\ dx = \int_0^{\frac{1}{2}} \sqrt{\frac{4x^2}{1 - 2x^2 + x^4} + 1}\ dx =$$

$$= \int_0^{\frac{1}{2}} \sqrt{\frac{4x^2 + 1 - 2x^2 + x^4}{1 - 2x^2 + x^4}}\ dx = \int_0^{\frac{1}{2}} \sqrt{\frac{1 + 2x^2 + x^4}{1 - 2x^2 + x^4}}\ dx = \int_0^{\frac{1}{2}} \sqrt{\frac{(1 + x^2)^2}{(1 - x^2)^2}}\ dx =$$

$$= \int_0^{\frac{1}{2}} \frac{1 + x^2}{1 - x^2}\ dx$$

Using long division, we find

$$L = \int_0^{\frac{1}{2}} -1 + \frac{1}{1 + x} + \frac{1}{1 - x}\ dx = [-x + \ln(1 + x) - \ln(1 - x)]_0^{\frac{1}{2}} = \ln(3) - \frac{1}{2}$$