THE JOHNS HOPKINS UNIVERSITY
Faculty of Arts and Sciences

MIDTERM EXAM - SPRING SESSION 2009

110.109 - CALCULUS II.

Examiner: Professor C. Consani
Duration: 50 MINUTES (10am-10:50am), April 24, 2009.

No calculators, books, notes allowed. **Total Points = 100**

Student Name: ______________________________

Ethic Stat.: I agree to complete this exam without unauthorized assistance from any person, materials or device.

Student Signature: ______________________________

TA Name (circle one): A. Banerjee, S. Khan, A. Saltz

Problem	Score
1	
2	
3	
4	
Total	

1.[25 points] For what values of p is the following integral finite? Fully explain your answer

$$\int_1^\infty x^p dx$$

Solution For $p \neq -1$ and for any $b > 1$, $b \in \mathbb{R}$

$$\int_1^b x^p dx = \frac{b^{p+1}}{p+1} - \frac{1}{p+1}$$

$\lim_{b\to\infty} \frac{b^{p+1}}{p+1} - \frac{1}{p+1} < \infty$ only if $p+1 \leq 0$. Since we have assumed $p \neq -1$, the inequality in fact means $p < -1$. The case $p = -1$ yields

$$\int_1^b x^p dx = \log b$$

$\lim_{b\to\infty} \log b = \infty$. Therefore the improper integral converges for $p < -1$.

2.[25 points] Determine if the following series are conditionally convergent, absolutely convergent, or divergent. Justify and explain your answer

(a)
$$\sum_{n=0}^{\infty} \frac{(-1)^n n^{2009}}{n!}$$

(a)
$$\sum_{n=1}^{\infty} \frac{2+\cos(n)}{n^2}$$

<u>Solution</u> (a) The factorial in the denominator suggests that the Ratio Test might be useful. We test it: let $a_n = \frac{(-1)^n n^{2009}}{n!}$, then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{(n+1)^{2009}}{(n+1)!}}{\frac{n^{2009}}{n!}} = \lim_{n\to\infty} \frac{(n+1)^{2009}}{(n+1)!} \cdot \frac{n!}{n^{2009}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{2009} \cdot \frac{n!}{(n+1)!} =$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{2009} \cdot \frac{1}{n+1} = 0 < 1$$

By the Ratio Test, this series converges absolutely.

(b) Recall that $-1 \leq \cos(n) \leq 1$, so $\dfrac{2+\cos(n)}{n^2} \leq \dfrac{3}{n^2}$ for all $n \geq 1$.
$\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2} = 3\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$, so it converges. By the Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{2+\cos(n)}{n^2}$ is also convergent. All the terms in these series are positive, so we can just say "convergent" (absolutely convergent is technically true as well).

3.[25 points]

(a) Compute the MacLaurin series of $f(x) = xe^x$ (i.e. the Taylor series at $x = 0$).

(b) Integrate the series in (a) from 0 to 1 to show that

$$\sum_{k=0}^{\infty} \frac{1}{(k+2)k!} = 1.$$

(You may assume that the MacLaurin series of $f(x)$ coincides with $f(x)$ on $[0, 1]$).

Solution (a) This can be done two ways. First, we recall that the Taylor series of $g(x) = e^x$ at $x = 0$ is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

So the MacLaurin series of $f(x) = xe^x$ is

$$x \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k!}.$$

Alternatively, we can compute the Taylor series of $f(x)$ at $x = 0$ directly:

$$\begin{aligned}
f(x) &= 0 \\
f'(x) &= xe^x + e^x \\
f''(x) &= (xe^x + e^x)' = xe^x + e^x + e^x = xe^x + 2e^x \\
f'''(x) &= (xe^x + 2e^x)' = xe^x + e^x + 2e^x = xe^x + 3e^x.
\end{aligned}$$

It follows that $f^{(k)}(x) = xe^x + ke^x$ and that in particular, $f^{(k)}(0) = k$. Therefore

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)x^k}{k!} = \sum_{k=0}^{\infty} \frac{kx^k}{k!} = \sum_{k=1}^{\infty} \frac{x^k}{(k-1)!} = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k!}$$

is the MacLaurin series for $f(x)$.

(b) It follows from (a) that $xe^x = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k!}$, at least on $[0, 1]$. So

$$\int_0^1 xe^x dx = \int_0^1 \sum_{k=0}^{\infty} \frac{x^{k+1}}{k!}.$$

Also, by applying the technique of integration by parts with $u = x$, $dv = e^x$

$$\int_0^1 xe^x dx = xe^x|_0^1 - \int_0^1 e^x dx = e - 0 - (e - 1) = 1.$$

On the other hand

$$\int_0^1 \sum_{k=0}^{\infty} \frac{x^{k+1}}{k!} dx = \sum_{k=0}^{\infty} \int_0^1 \frac{x^{k+1}}{k!} dx = \sum_{k=0}^{\infty} \frac{x^{k+2}}{(k+2)k!}\Big|_{x=0}^{x=1} = \sum_{k=0}^{\infty} \frac{(1)^{k+2}}{(k+2)k!} = \sum_{k=0}^{\infty} \frac{1}{(k+2)k!}.$$

4.[25 points] Determine the largest interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{\arctan |x|^n}{n + |\sin(nx)|}$$

$(\arctan(x) = \tan^{-1}(x)$. You may use the inequality: $\arctan |x| \leq |x|)$

Solution We apply the Comparison Test

$$\frac{\arctan |x|^n}{n + |\sin(nx)|} \leq \frac{|x|^n}{n}$$

and study the convergence of the series $\sum_{n\geq 1} \frac{|x|^n}{n}$. To this series we apply the Root Test

$$\lim_{n\to\infty} \frac{|x|}{\sqrt[n]{n}} = |x|$$

Then, the above series (and hence the original one) is convergent for $|x| < 1$.
At $|x| = 1$, $\lim_{n\to\infty} \frac{\frac{\pi}{4}}{n + |\sin(n)|} = 0$, however $\frac{\frac{\pi}{4}}{n + |\sin(n)|} \geq \frac{\frac{\pi}{4}}{n+1}$ and $\sum_{n\geq 1} \frac{\frac{\pi}{4}}{n+1} = \infty$. Therefore by applying the Comparison Test we conclude that the original series is divergent at $|x| = 1$.

When $|x| \geq 1$, we notice that $\frac{\frac{\pi}{4}}{n+1} \leq \frac{\arctan |x|^n}{n + |\sin(nx)|}$, and the series $\sum_{n\geq 1} \frac{\frac{\pi}{4}}{n+1}$ is divergent. Then, we conclude that the largest interval of convergence of the original series is $(-1, 1)$.