Problem 1. Compute the antiderivative

$$\int e^x \cos x \, dx.$$

$$\int e^x \cos x\, dx = e^x \cos x + \int e^x \sin x\, dx$$ 5

$$= e^x \cos x + e^x \sin x - \int e^x \cos x\, dx$$ 5

$$\Rightarrow \int e^x \cos x\, dx = \frac{1}{2} e^x (\cos x + \sin x) + C$$ 5

$$\left(\frac{1}{2} e^x (\cos x + \sin x)\right)' = \frac{1}{2} e^x (\cos x + \sin x) + \frac{1}{2} e^x (-\sin x + \cos x)$$

$$= e^x \cos x$$

$u = \cos x \Rightarrow du = -\sin x\, dx$

$dv = e^x dx \Rightarrow v = e^x$

$\cos x\, e^x + \int e^x \sin x$

$du = \cos x\, dx \Rightarrow u = \sin x$

$v = e^x \Rightarrow dv = e^x$

$e^x \sin x - \int \sin x\, e^x\, dx$

Problem 2. Compute the antiderivative

$$\int \sin^2 x \cos^5 x \, dx.$$

$$\int \sin^2 x \cos^5 x \, dx = \int \sin^2 x (1-\sin^2 x)^2 \cos x \, dx$$

$$= \int u^2 (1-u^2)^2 \, du$$

$$= \int u^6 - 2u^4 + u^2 \, du$$

$$= \frac{u^7}{7} - \frac{2}{5}u^5 + \frac{u^3}{3} + C$$

$$= \frac{\sin^7 x}{7} - \frac{2}{5}\sin^5 x + \frac{\sin^3 x}{3} + C$$

Problem 3. Compute the antiderivative

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx,$$

where a is a constant real number. (Your answer will be in terms of a.)

$\sin^2\theta+\cos^2\theta=1$

$\tan^2\theta+1=\sec^2\theta$

$\tan^2\theta=\sec^2\theta-1$

$a^2\tan^2\theta=a^2\sec^2\theta-a^2$

$x=a\sec\theta$

$dx=a\sec\theta\tan\theta\,d\theta$

$u=\sec\theta+\tan\theta$

$du=(\sec\theta\tan\theta+\sec^2\theta)\,d\theta$

$$\int\frac{1}{\sqrt{x^2-a^2}}\,dx=\int\frac{a\sec\theta\tan\theta\,d\theta}{\tan\theta}$$

$$=\int a\sec\theta\,d\theta$$

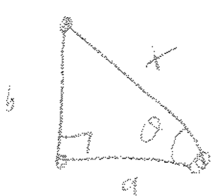
$$=a\int\frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\,d\theta$$

$$=a\int\frac{du}{u}$$

$$=a\ln|u|+C$$

$$=a\ln|\sec\theta+\tan\theta|+C$$

$$=a\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+C$$



$\sec\theta=\frac{\text{hyp}}{\text{adj}}$

$\tan\theta=\frac{\text{opp}}{\text{adj}}=\frac{\sqrt{x^2-a^2}}{a}$

$a^2+b^2=x^2$

$b=\sqrt{x^2-a^2}$

Problem 4. Compute the antiderivative

$$\int \frac{x^3}{(x-1)^2}\,dx.$$

$y = x-1$

$dy = dx$

$$\int \frac{x^3}{(x-1)^2}\,dx = \int \frac{(y+1)^3}{y^2}\,dy \qquad \text{5pts}$$

$$= \int \frac{y^3+3y^2+3y+1}{y^2}\,dy \qquad \text{1pts}$$

$$= \int y + 3 + \frac{3}{y} + \frac{1}{y^2}\,dy$$

$$= \frac{y^2}{2} + 3y + 3\ln|y| - \frac{1}{y} + C$$

$$= \underbrace{\frac{(x-1)^2}{2}}_{\text{3pts}} + \underbrace{3(x-1)}_{\text{3pts}} + \underbrace{3\ln|x-1|}_{\text{3pts}} - \underbrace{\frac{1}{x-1}}_{\text{3pts}} + \underbrace{C}_{\text{2pts}}$$

5pts

$$\begin{array}{r|l} & x+2 \quad \text{r } 3x-2 \\ x^2-2x+1 & x^3+0x^2+0x+0 \\ & x^3-2x^2+x \\ & 2x^2-x+0 \\ & 2x^2-4x+2 \\ & 3x-2 \end{array}$$

$$I = \int x+2+\frac{3x-2}{(x-1)^2}\,dx$$

$$= \frac{x^2}{2} + 2x + \int \frac{3x-2}{x^2-2x+1}\,dx \qquad \text{3pts}$$

$$= \frac{x^2}{2} + 2x + \int \frac{3x-2}{(x-1)^2}\,dx$$

$x = u+1$

$u = x-1$

$du = dx$

$$= \frac{x^2}{2} + 2x + \int \frac{3u+1}{u^2}\,du \qquad \text{3pts}$$

$$= \frac{x^2}{2} + 2x + \int 3u^{-1} + u^{-2}\,du \qquad \text{3pts}$$

$$= \frac{x^2}{2} + 2x + 3\ln|u| - u^{-1} + C \qquad \text{3pts}$$

$$= \frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1} + C \qquad \text{3pts}$$

$u = \ln x \quad dv = dx$
$du = \frac{1}{x}dx \quad v = x$

$\int \ln x \, dx$
$= x\ln x - \int \frac{x}{x} dx$
$= x\ln x - x + C$

Problem 5. Solve the differential equation

$$\frac{dy}{dx} + y \ln x = \frac{1}{x^x},$$

where x is restricted to the positive real numbers (i.e., $x > 0$).
(Hint: $\frac{1}{x^x} = e^{-x \ln x}$.)

$I(x) = \exp\left(\int \ln x \, dx\right) = \exp(x\ln x - x)$

$\left(e^{x\ln x - x} y\right)' = \frac{1}{x^x} e^{x\ln x - x} = \exp(x\ln x - x - x\ln x)$
$= e^{-x}$

$e^{x\ln x - x} y = -e^{-x} + C$
$y = -e^{-x\ln x} + Ce^{x - x\ln x}$

Problem 6. Compute the arclength of the parametric curve

$$\begin{aligned} x(t) &= at, \\ y(t) &= \frac{1}{3}t^3, \end{aligned}$$

from $t = 0$ to $t = 1$, where a is a constant real number. You answer will be in terms of a. (Hint: the formula

$$\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$$

may be useful to you.)

$$L = \int_0^1 \sqrt{(a)^2 + (t^2)^2}\, dt$$

$$= \int_0^1 \sqrt{a^2 + t^4}\, dt$$

↑

Elliptic integral! (Oops!)

+10 POINTS FOR $\int_0^1 \sqrt{x^4 + a^2}\, dx$

+5 POINTS FOR EACH CORRECT SUBSTITUTION THEREAFTER

Problem 7. Suppose a, b are constant real numbers. Find all real numbers θ such that the tangent line to the polar curve

$$r(\theta) = a + b\sin\theta$$

is horizontal at θ. (Your answer will be in terms of a and b.)

$$y = r\sin\theta = a\sin\theta + b\sin^2\theta$$

$$\frac{dy}{d\theta} = a\cos\theta + 2b\sin\theta\cos\theta$$
$$= \cos\theta(a + 2b\sin\theta)$$

$$x = r\cos\theta = a\cos\theta + b\sin\theta\cos\theta$$

$$\frac{dx}{d\theta} = -a\sin\theta + b\cos^2\theta - b\sin^2\theta$$

$$\frac{dy}{dx} = \frac{\cos\theta(a+2b\sin\theta)}{} = 0 \iff \cos\theta = 0 \text{ or } \sin\theta = \frac{-a}{2b}$$

$$\iff \theta = \frac{\pi}{2} + k\pi \text{ or } \theta = \sin^{-1}\left(\frac{-a}{2b}\right) + 2k\pi$$
for $k \in \mathbb{Z}$