**Name:** ______________________________ **Section Number:** ____________

# 110.109 CALCULUS II (Physical Sciences & Engineering)
# SPRING 2011
# MIDTERM EXAMINATION SOLUTIONS
# March 9, 2011

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**Instructions:** The exam is **7** pages long, including this title page. The number of points each problem is worth is listed after the problem number. The exam totals to one hundred points. For each item, please **show your work** or **explain** how you reached your solution. Please do all the work you wish graded on the exam. Good luck !!

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## PLEASE DO NOT WRITE ON THIS TABLE !!

| Problem | Score | Points for the Problem |
|---|---|---|
| 1 | | 20 |
| 2 | | 20 |
| 3 | | 15 |
| 4 | | 15 |
| 5 | | 30 |
| TOTAL | | 100 |

### Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

**Signature:** ______________________________ **Date:** ____________

**Question 1.** [20 points] Do the following:

**(a)** Find the value of $\displaystyle\int_0^{\frac{\pi}{4}} \tan^3\theta \sec^4\theta \, d\theta$.

**Solution: Two things to notice first: (1) $\tan^2\theta + 1 = \sec^2\theta$, and (2) $\frac{d}{d\theta}\tan\theta = \sec^2\theta$. Thus we can write**

$$\int_0^{\frac{\pi}{4}} \tan^3\theta \sec^4\theta \, d\theta = \int_0^{\frac{\pi}{4}} \tan^3\theta \sec^2\theta \sec^2\theta \, d\theta = \int_0^{\frac{\pi}{4}} \tan^3\theta \left(\tan^2\theta + 1\right) \sec^2\theta \, d\theta.$$

**We use the substitution $x = \tan\theta$, $dx = \sec^2\theta \, d\theta$ (along with the limits: when $\theta = 0$, $x = 0$, and when $\theta = \frac{\pi}{4}$, we have $x = 1$), and get**

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \tan^3\theta \left(\tan^2\theta + 1\right) \sec^2\theta \, d\theta &= \int_0^1 x^3 \left(x^2 + 1\right) \, dx \\ &= \int_0^1 \left(x^5 + x^3\right) \, dx = \int_0^1 \left(\frac{x^6}{6} + \frac{x^4}{4}\right)\Bigg|_0^1 = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}.\end{aligned}$$

**(b)** Find an antiderivative of the function $f(x) = \dfrac{x-1}{(x-2)(x-3)}$.

**Solution: This question asks to calculate $\displaystyle\int \frac{x-1}{(x-2)(x-3)} \, dx$. To this end, decompose the rational function $f(x)$ into the sum two simpler rational functions:**

$$\frac{x-1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3},$$

**where the two constants $A$ and $B$ are unknown. Recombining the fractions on the right, we get**

$$\begin{aligned}\frac{x-1}{(x-2)(x-3)} &= \frac{A}{x-2} + \frac{B}{x-3} \\ &= \frac{A(x-3)}{(x-2)(x-3)} + \frac{B(x-2)}{(x-2)(x-3)} \\ &= \frac{Ax - 3A + Bx - 2B}{(x-2)(x-3)} = \frac{(A+B)x + (-3A-2B)}{(x-2)(x-3)}.\end{aligned}$$

**Equating the rational functions on the ends of this last set of equations yields two equations in the two unknowns:**

$$\begin{aligned}A + B &= 1 \\ -3A - 2B &= -1.\end{aligned}$$

**Solving these yields $A = -1$ and $B = 2$. Thus**

$$\begin{aligned}\int \frac{x-1}{(x-2)(x-3)} \, dx &= \int \left(\frac{-1}{x-2} + \frac{2}{x-3}\right) dx \\ &= -\int \frac{1}{x-2} \, dx + 2\int \frac{1}{x-3} \, dx = -\ln|x-2| + 2\ln|x-3| + C.\end{aligned}$$

**Question 2.** [20 points] Solve exactly one of the following 2 first-order differential equations (you must choose which one is your solution by circling either **(a)** or **(b)**):

**(a)** $xy' = x\ln x + y, \quad y(e) = 0.$

**Solution:** **This differential equation is linear (though not separable). Written in standard form, we get $y' - \frac{1}{x}y = \ln x$. The integrating factor is**

$$e^{-\int \frac{1}{x}\,dx} = e^{-\ln x} = e^{\ln x^{-1}} = e^{\ln\frac{1}{x}} = \frac{1}{x}.$$

**Multiply through the differential equation to get**

$$\frac{1}{x}y' - \frac{1}{x^2}y = \frac{\ln x}{x} \qquad \textbf{or} \qquad \frac{d}{dx}\left[\frac{1}{x}y\right] = \frac{\ln x}{x}.$$

**Integrate both sides with respect to $x$. The left hand side is straightforward (antiderivative of a derivative). To get the right hand side, use the substitution $u = \ln x$, $du = \frac{1}{x}\,dx$. So**

$$\int \frac{d}{dx}\left[\frac{1}{x}y\right]dx = \int \frac{\ln x}{x}\,dx$$

$$\frac{1}{x}y = \int u\,du + C = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C.$$

**Solving for $y$, we finally get**

$$y(x) = \frac{x(\ln x)^2}{2} + Cx.$$

**And with the initial data,**

$$y(e) = \frac{e(\ln e)^2}{2} + Ce = \frac{e}{2} + Ce = 0.$$

**This is solved by $C = -\frac{1}{2}$. Our final solution is**

$$y(x) = \frac{x(\ln x)^2}{2} - \frac{x}{2} = \frac{x}{2}\left((\ln x)^2 - 1\right).$$

**(b)** $x^2yy' + y = y^2, \quad y(1) = 0.$

**Solution:** **This differential equation is separable (and also linear!).**

**METHOD 1 Linear:** **Divide the entire equation by the common $y$ and manipulate to put it in standard form to get $y' - \frac{1}{x^2}y = -\frac{1}{x^2}$. The integrating factor is**

$$e^{-\int \frac{1}{x^2}\,dx} = e^{\frac{1}{x}},$$

**and upon multiplication, we get**

$$e^{\frac{1}{x}}\left(y' - \frac{1}{x^2}y = -\frac{1}{x^2}\right) \qquad \textbf{or} \qquad e^{\frac{1}{x}}y' - e^{\frac{1}{x}}\frac{1}{x^2}y = -e^{\frac{1}{x}}\frac{1}{x^2} \qquad \textbf{or} \qquad \frac{d}{dx}\left[e^{\frac{1}{x}}y\right] = -e^{\frac{1}{x}}\frac{1}{x^2}.$$

**Integrate both sides with respect to $x$. The left hand side is straightforward (antiderivative of a derivative). To get the right hand side, use the substitution $u = \frac{1}{x}$, $du = -\frac{1}{x^2}\,dx$. So**

$$\int \frac{d}{dx}\left[e^{\frac{1}{x}}y\right]dx = \int -e^{\frac{1}{x}}\frac{1}{x^2}\,dx$$

$$e^{\frac{1}{x}}y = \int e^u\,du + C = e^u + C = e^{\frac{1}{x}} + C.$$

**Solving for $y$, we finally get**

$$y(x) = 1 + Ce^{-\frac{1}{x}}.$$

**And with the initial data,**

$$y(1) = 1 + Ce^{-\frac{1}{1}} = 1 + \frac{C}{e} = 0.$$

**This is solved by $C = -e$, and our final solution is $y(x) = 1 + e^{-1}e^{-\frac{1}{x}} = 1 + e^{-\left(1+\frac{1}{x}\right)}$.**

**METHOD 2 Separable:** Divide the entire equation by the common $y$ and manipulate to sole for $y'$. We get $y' = \frac{y-1}{x^2}$. Written in differential form (to prepare for integration), we have

$$\frac{1}{y-1}\,dy = \frac{1}{x^2}\,dx.$$

Integrate each side with respect to its only variable to get

$$\int \frac{1}{y-1}\,dy = \int \frac{1}{x^2}\,dx \qquad \text{or} \qquad \ln|y-1| = -\frac{1}{x} + C.$$

Whenever two expressions are equal, the exponentials of the expressions are also equal, and

$$e^{\ln|y-1|} = e^{-\frac{1}{x}+C} \qquad \text{or} \qquad |y-1| = e^{-\frac{1}{x}}e^{C} \qquad \text{or} \qquad y-1 = Ke^{-\frac{1}{x}},$$

where we can relax the absolute values by allowing the constant to assume negative values. And finally, solving for $y$, we get

$$y(x) = 1 + Ke^{-\frac{1}{x}}.$$

Note this is the same solution as in Method 1 above. And with the initial data,

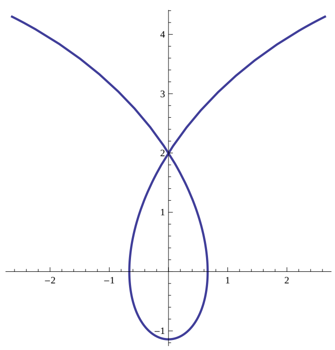
$$y(1) = 1 + Ke^{-\frac{1}{1}} = 1 + \frac{K}{e} = 0.$$

This is solved by $K = -e$, and our final solution is $y(x) = 1 + e^{-1}e^{-\frac{1}{x}} = 1 + e^{-\left(1+\frac{1}{x}\right)}$.

**Question 3.** [15 points] For the parameterization

$$x(t) = 2t - \pi \sin t$$
$$y(t) = 2 - \pi \cos t,$$

find the equations of all tangent lines to the curve at the point $(0, 2)$.




**Solution:** First, we need to find the values of $t$ that correspond to the point $(x, y) = (0, 2)$. For $y(t) = 2$, we will need to solve $y(t) = 2 - \pi\cos t = 2$. This means $\cos t = 0$ which is solved by $t = \pm\frac{\pi}{2}$. Back in the equation for $x$, we find that both

$$x(t)\Big|_{t=\frac{\pi}{2}} = x\left(\frac{\pi}{2}\right) = 2\left(\frac{\pi}{2}\right) - \pi\sin\left(\frac{\pi}{2}\right) = \pi - \pi = 0, \qquad \text{and}$$
$$x(t)\Big|_{t=-\frac{\pi}{2}} = x\left(-\frac{\pi}{2}\right) = 2\left(-\frac{\pi}{2}\right) - \pi\sin\left(-\frac{\pi}{2}\right) = -\pi + \pi = 0.$$

Now, for $t = \frac{\pi}{2}$, we get

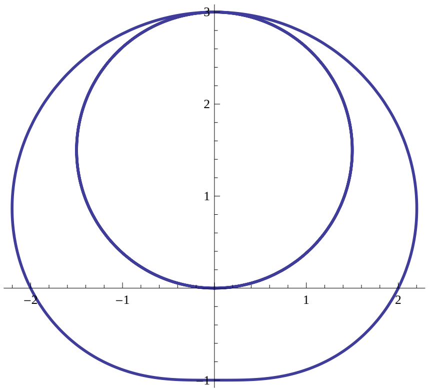
$$\frac{dy}{dx}\Bigg|_{(x,y)=(0,2)} = \frac{\frac{dy}{dt}\Big|_{t=\frac{\pi}{2}}}{\frac{dx}{dt}\Big|_{t=\frac{\pi}{2}}} = \frac{(\pi\sin t)\Big|_{t=\frac{\pi}{2}}}{(2-\pi\cos t)\Big|_{t=\frac{\pi}{2}}} = \frac{\pi}{2-0} = \frac{\pi}{2}.$$

Hence, using the point-slope form for the equation of a line, we get

$$y - 2 = \frac{\pi}{2}(x - 0) \qquad \text{or} \qquad y = \frac{\pi}{2}x + 2.$$

Note that the other equation is almost exactly the same (the slope includes a minus sign), and we get $y = -\frac{\pi}{2}x + 2$.

**Question 4.** [15 points] Find the total area inside of the polar curve $r = 2 + \sin\theta$ and outside the polar curve $r = 3\sin\theta$.




**Solution:** **First, Note here that there is only one intersection point between these two curves, at $\theta = \frac{\pi}{2}$. Hence one curve always lives "inside" the other, and integration is only about finding the area of each and subtracting. In this case, the "inside" curve is the circle, or the one-leaf rose $r = 3\sin\theta$. Hence we find the area inside each curve using the formula**

$$\textbf{Area} = \int_{\alpha}^{\beta} \frac{1}{2}\left[f(\theta)\right]^2\, d\theta,$$

**and subtract the area inside the outside curve from that of the inside curve. Note that the limits in this area formula must be found so that we trace out the curve exactly once in each case. The formula for the area we seek is**

$$\textbf{Area between curves} = \int_0^{2\pi} \frac{1}{2}\left[2+\sin\theta\right]^2\, d\theta - \int_0^{\pi} \frac{1}{2}\left[3\sin\theta\right]^2\, d\theta.$$

**Here are some of the (overly) detailed calculations:**

$$\begin{aligned}
\textbf{Area between curves} &= \int_0^{2\pi} \frac{1}{2}\left[2+\sin\theta\right]^2\, d\theta - \int_0^{\pi} \frac{1}{2}\left[3\sin\theta\right]^2\, d\theta \\
&= \int_0^{2\pi} \frac{1}{2}\left(4 + 4\sin\theta + \sin^2\theta\right)\, d\theta - \int_0^{\pi} \frac{9}{2}\sin^2\theta\, d\theta \\
&= \int_0^{2\pi} \left(2 + 2\sin\theta + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)\right)\, d\theta - \int_0^{\pi} \frac{9}{2}\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)\, d\theta \\
&= \int_0^{2\pi} \left(2 + 2\sin\theta + \frac{1}{4} + \frac{1}{4}\cos 2\theta\right)\, d\theta - \int_0^{\pi} \left(\frac{9}{4} + \frac{9}{4}\cos 2\theta\right)\, d\theta \\
&= \left(2\theta - 2\cos\theta + \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta\right)\Big|_0^{2\pi} - \left(\frac{9}{4}\theta + \frac{9}{8}\sin 2\theta\right)\Big|_0^{\pi} \\
&= \left(\left(2(2\pi) - 2\cos 2\pi + \frac{1}{4}(2\pi) + \frac{1}{8}\sin 2(2\pi)\right) - \left(2(0) - 2\cos 0 + \frac{1}{4}(0) + \frac{1}{8}\sin 2(0)\right)\right) \\
&\quad - \left(\left(\frac{9}{4}\pi + \frac{9}{8}\sin 2\pi\right) - \left(\frac{9}{4}(0) + \frac{9}{8}\sin 2(0)\right)\right) \\
&= \left(\frac{9}{2}\pi - 2\right) + 2 - \frac{9}{4}\pi = \frac{9}{4}\pi.
\end{aligned}$$

**Notice that the area inside the outside curve is $\frac{9}{2}\pi$, and the area inside the inside curve (the circle) is $\frac{9}{4}\pi$. Further noticing that the inside curve is a circle, you could have appealed directly to geometry, and said that the radius of the inside curve is $\frac{3}{2}$, so that the area inside the circle is $\pi\left(\frac{3}{2}\right)^2 = \frac{9}{4}\pi$.**

**Question 5.** [30 points] Given the parameterization $x(t) = \frac{2}{3}t^3 + 1$ and $y(t) = 3 - t^2$ on the interval $0 \le t \le 1$, do the following:

**(a)** Calculate the total arc-length of the curve.

**Solution:** **The formula for the arc-length of a parameterized curve is**

$$\textbf{Arc-length} = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

**In our case, $\alpha = 0$, $\beta = 1$, and the formula is**

$$\begin{aligned}\textbf{Arc-length} &= \int_0^1 \sqrt{(2t^2)^2 + (-2t)^2}\, dt \\ &= \int_0^1 \sqrt{4t^4 + 4t^2}\, dt \\ &= \int_0^1 2t\sqrt{t^2+1}\, dt.\end{aligned}$$

**This last integral is perfectly set up for a standard substitution of $u = t^2 + 1$, $du = 2t\,dt$, with limits: When $t = 0$, $u = 1$, and when $t = 1$, $u = 2$. We get**

$$\textbf{Arc-length} = \int_0^1 2t\sqrt{t^2+1}\, dt = \int_1^2 \sqrt{u}\, du = \left.\frac{1}{2\sqrt{u}}\right|_1^2 = \frac{1}{2\sqrt{2}} - \frac{1}{2}.$$

**Simplify if you want, but that is the answer.**

**(b)** Find the area between the curve and the $x$-axis.

**Solution:** **The formula for the area "under" a parameterized curve is**

$$\textbf{Area} = \int_\alpha^\beta y(t)x'(t)\, dt.$$

**In our case again, $\alpha = 0$, $\beta = 1$, and the formula is**

$$\textbf{Area} = \int_0^1 (3 - t^2)(2t^2)\, dt = \int_0^1 \left(6t^2 - 2t^4\right)\, dt = \left.\left(\frac{6}{3}t^3 - \frac{2}{5}t^5\right)\right|_0^1 = 2 - \frac{2}{5} = \frac{8}{5}.$$

**Simplify if you want, but that is the answer.**

## Possibly helpful formulae

- $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$
- $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$
- $\sin^2\theta + \cos^2\theta = 1$
- $\tan^2\theta + 1 = \sec^2\theta$
- $\sin 2\theta = 2\cos\theta\sin\theta$
- $\cos 2\theta = \cos^2\theta - \sin^2\theta$
- $\sin(-\theta) = -\sin\theta$
- $\cos(-\theta) = \cos\theta$

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| $\theta$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
|---|---|---|---|
| $0$ | $0$ | $1$ | $0$ |
| $\frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{3}}$ |
| $\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | $1$ |
| $\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |
| $\frac{\pi}{2}$ | $1$ | $0$ | $-$ |