Name: ____ **Section Number:** ______________

110.109 CALCULUS II (Physical Sciences & Engineering)
SPRING 2011
MIDTERM EXAMINATION SOLUTIONS
May 4, 2011

Instructions: The exam is **7** pages long, including this title page. The number of points each problem is worth is listed after the problem number. The exam totals to one hundred points. For each item, please **show your work** or **explain** how you reached your solution. Please do all the work you wish graded on the exam. Good luck !!

PLEASE DO NOT WRITE ON THIS TABLE !!

Problem	Score	Points for the Problem
1		10
2		20
3		15
4		15
5		20
6		20
TOTAL		100

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ____ **Date:** ______________

Question 1. [10 points] Solve exactly one of the following 2 improper integral problems (you must choose which one is your solution by circling either **(a)** or **(b)**):

(a) Calculate (or show it does not exist) $\displaystyle\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx.$

Solution: **This integral is improper since the integrand is not defined at the lower limit. So first, we correct the problem, and write** $\displaystyle\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$. **Now we can integrate. Notice that the integrand has a structure easily simplified with the substitution** $u=\sqrt{x}$, **and** $du = \frac{1}{2\sqrt{x}}\,dx$, **so that** $2\,du = \frac{1}{\sqrt{x}}\,dx$. **We can also change the limits immediately: When** $x=a$, $u=\sqrt{a}$ **and when** $x=1$, $u=1$. **We get:**

$$
\begin{aligned}
\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx &= \lim_{a\to 0^+}\int_a^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx \\
&= \lim_{a\to 0^+}\int_{\sqrt{a}}^1 2e^{-u}\,dx \\
&= \lim_{a\to 0^+}\left(-2e^{-u}\right)\Big|_{\sqrt{a}}^1 \\
&= \lim_{a\to 0^+} -2e^{-1}+2e^{-\sqrt{a}} = -2e^{-1}+2 = 2-\frac{2}{e}.
\end{aligned}
$$

This is like Exercise 14, Section 7.8 but with different limits.

(b) Calculate (or show it does not exist) $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^{\frac{3}{2}}}\,dx.$

Solution: **This integral is improper since the interval of integration is of infinite length. We correct the problem by writing** $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^{\frac{3}{2}}}\,dx = \lim_{b\to\infty}\int_e^b \frac{1}{x(\ln x)^{\frac{3}{2}}}\,dx$. **The integral inside the limit is now a simple definite integral. The integrand again has the structure easily simplified with a substitution;** $u=\ln x$, **and** $du=\frac{1}{x}\,dx$. **We also change the limits immediately: When** $x=e$, $u=\ln e = 1$ **and when** $x=b$, $u=\ln b$. **We get:**

$$
\begin{aligned}
\int_e^\infty \frac{1}{x(\ln x)^{\frac{3}{2}}}\,dx &= \lim_{b\to\infty}\int_e^b \frac{1}{x(\ln x)^{\frac{3}{2}}}\,dx \\
&= \lim_{b\to\infty}\int_1^{\ln b} \frac{1}{u^{\frac{3}{2}}}\,dx \\
&= \lim_{b\to\infty}\left(-\frac{2}{u^{\frac{1}{2}}}\right)\Big|_1^{\ln b} \\
&= \lim_{b\to\infty} -\frac{2}{(\ln b)^{\frac{1}{2}}} + \frac{2}{(1)^{\frac{1}{2}}} + = -2.
\end{aligned}
$$

This is like Exercise 25, Section 7.8 but with an exponent of $\frac{3}{2}$ instead of 3.

Question 2. [20 points] Determine if the following converge or not:

(a) $\left\{\dfrac{\ln n}{\sqrt{n}}\right\}_{n=1}^{\infty}$

Solution: **It is not *a priori* clear where this sequence may go, since as n goes to ∞, both the numerator and the denominator both go to ∞. However, both "look" like differentiable functions, so we can use a function to study this sequence. Let $f(x) = \frac{\ln x}{\sqrt{x}}$, so that $f(n) = \frac{\ln n}{\sqrt{n}}$ is our sequence. This sequence has a limit if $f(x)$ has a horizontal asymptote. Hence we study**

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}.$$

Both the numerator and denominator (1) go to infinity, and (2) are differentiable, so by L'Hospital's Rule,

$$\begin{aligned}\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} &= \lim_{x\to\infty} \frac{\frac{d}{dx}[\ln x]}{\frac{d}{dx}[\sqrt{x}]}\\ &= \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}\\ &= \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.\end{aligned}$$

Hence $\left\{\dfrac{\ln n}{\sqrt{n}}\right\}_{n=1}^{\infty}$ converges to 0. This problem is much like Exercises 32 and 42, Section 11.1 in notation and method for solution.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{3n^2+6}{2n^4-n^3-2}$

Solution: **Think of the terms of this series as $a_n = \frac{3n^2+6}{2n^4-n^3-2}$. We would love to be able to compare this series, term by term, directly with a series whose terms are $b_n = \frac{3n^2}{2n^4}$, since**

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{3n^2}{2n^4} = \left(\frac{3}{2}\right)\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a p-series with $p = 2$, and hence converges. However, comparing a_n and b_n directly, we see that the numerator of a_n is larger than that of b_n, while the denominator of a_n is smaller than that of b_n. Hence $a_n > b_n$, and we cannot compare use the Comparison Test directly. Instead, we use the Limit Comparison Test: Here,

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{3n^2+6}{2n^4-n^3-2}}{\frac{3n^2}{2n^4}} = \lim_{n\to\infty} \left(\frac{3n^2+6}{2n^4-n^3-2}\cdot\frac{2n^4}{3n^2}\right) = \lim_{n\to\infty} \frac{6n^6+12n^4}{6n^6-3n^5-2n^2}\\ &= \lim_{n\to\infty} \frac{\left(6n^6+12n^4\right)\left(\frac{1}{n^6}\right)}{\left(6n^6-3n^5-2n^2\right)\left(\frac{1}{n^6}\right)} = \lim_{n\to\infty} \frac{6+\frac{12}{n^2}}{6-\frac{3}{n}-\frac{2}{n^4}} = 1.\end{aligned}$$

Since this limit exists and is not 0, we can conclude that both series either converge or diverge. And since $\sum b_n$ converges as a p-series with $p = 2$, we conclude that the original series also converges. This problem is much like Exercise 24, Section 11.4.

Question 3. [15 points] Use the geometric series $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ to find a power series representation for $\displaystyle\int \frac{x^2}{1+2x}\,dx$.

Solution: First we manipulate the integrand and write it as a power series:

$$\frac{x^2}{1+2x} = x^2\left(\frac{1}{1+2x}\right) = x^2\left(\frac{1}{1-(-2x)}\right).$$

Now we can use the form of the geometric series directly:

$$\frac{x^2}{1+2x} = x^2\left(\frac{1}{1-(-2x)}\right) = x^2\sum_{n=0}^{\infty}(-1)^n 2^n x^n = \sum_{n=0}^{\infty}(-1)^n 2^n x^{n+2}.$$

Now all that it left is to integrate the power series, which we do term by term:

$$\int \frac{x^2}{1+2x}\,dx = \int\left(\sum_{n=0}^{\infty}(-1)^n 2^n x^{n+2}\right)dx = C + \sum_{n=0}^{\infty}(-1)^n 2^n \frac{x^{n+3}}{n+3}.$$

We are done. This problem is similar to Exercise 23, Section 11.9 in its notation and method for solution.

Question 4. [15 points] Calculate the Taylor series for $h(x) = (1+2x)^3$ at $x = 1$.
(Hint: $h(x)$ is a polynomial.)

Solution: **First note that this is will not be an infinite binomial series since $h(x)$ is a polynomial, and we are not looking for the Maclauren series. The form for a binomial series assumes $a = 0$, while here we have $a = 1$. It is true that, since $h(x)$ is a polynomial, so will be its Taylor series (and of the same degree!). We start by calculating derivatives:**

$$\begin{aligned}
h(1) &= (1+2x)^3\Big|_{x=1} = 27 \\
h'(1) &= 3(1+2x)^2 2\Big|_{x=1} = 6(1+2x)^2\Big|_{x=1} = 54 \\
h''(1) &= 12(1+2x)2\Big|_{x=1} = 24(1+2x)\Big|_{x=1} = 72 \\
h^{(3)}(1) &= 24 \cdot 2 = 48 \\
h^{(n)}(1) &= 0 \textbf{ for } n \geq 4.
\end{aligned}$$

Now we write the series:

$$\begin{aligned}
h(x) &= \sum_{n=0}^{\infty} \frac{h^{(n)}(1)}{n!}(x-1)^n \\
&= \frac{h(1)}{0!}(x-1)^0 + \frac{h'(1)}{1!}(x-1) + \frac{h''(1)}{2!}(x-1)^2 + \frac{h^{(3)}(1)}{3!}(x-1)^3 + \frac{h^{(4)}(1)}{4!}(x-1)^4 + \dots \\
&= 27 + 54(x-1) + \frac{72}{2!}(x-1)^2 + \frac{48}{3!}(x-1)^3 \\
&= 27 + 54(x-1) + 36(x-1)^2 + 8(x-1)^3.
\end{aligned}$$

This problem is almost the same as Exercise 14, Section 11.10, but with a different polynomial and value for a.

Question 5. [20 points] Suppose the Taylor series of a function $g(x)$ was determined to be

$$\sum_{n=1}^{\infty} \frac{2^n}{n}(x-2)^n.$$

Calculate the interval of convergence for this series (that is, find the set of all values of x where $g(x) = \sum_{n=1}^{\infty} \frac{2^n}{n}(x-2)^n$).

Solution: First we find the radius of convergence R for this series, and use the Ratio Test for this. Here $a_n = \frac{2^n}{n}(x-2)^n$ and $a_{n+1} = \frac{2^{n+1}}{n+1}(x-2)^{n+1}$, so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{n+1}(x-2)^{n+1}}{\frac{2^n}{n}(x-2)^n}\right| = \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)|x-2| = 2|x-2|\lim_{n\to\infty}\frac{n}{n+1}.$$

The Ratio Test concludes that the series will converge as long as this limit is less than 1. Hence we solve the inequality

$$2|x-2|\lim_{n\to\infty}\frac{n}{n+1} = 2|x-2| < 1,$$

which leads to $|x-2| < \frac{1}{2}$ and hence our radius of convergence is $R = \frac{1}{2}$. It is also possible that the series will converge when x is chosen so that the Ratio Test limit equals 1. This happens when

$$|x-2| = \frac{1}{2} \text{ or when } x = \frac{3}{2} \text{ and } x = \frac{5}{2}.$$

We test these individually. First, let $x = \frac{5}{2}$. Then

$$\left(\sum_{n=1}^{\infty}\frac{2^n}{n}(x-2)^n\right)\Bigg|_{x=\frac{5}{2}} = \sum_{n=1}^{\infty}\frac{2^n}{n}\left(\frac{5}{2}-2\right)^n = \sum_{n=1}^{\infty}\frac{2^n}{n}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty}\frac{2^n}{n}\frac{1}{2^n} = \sum_{n=1}^{\infty}\frac{1}{n}$$

which diverges since it is the Harmonic series (hence $x = \frac{5}{2}$ is not in the interval of convergence). For $x = \frac{3}{2}$, we get

$$\left(\sum_{n=1}^{\infty}\frac{2^n}{n}(x-2)^n\right)\Bigg|_{x=\frac{3}{2}} = \sum_{n=1}^{\infty}\frac{2^n}{n}\left(\frac{3}{2}-2\right)^n = \sum_{n=1}^{\infty}\frac{2^n}{n}\left(\frac{-1}{2}\right)^n = \sum_{n=1}^{\infty}\frac{2^n}{n}\frac{(-1)^n}{2^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

which converges as the Alternating Harmonic series (hence $x = \frac{3}{2}$ IS included in the interval of convergence). Hence the interval of convergence

$$\left[\frac{3}{2}, \frac{5}{2}\right).$$

This problem is close to Exercise 18 Section 11.8.

Question 6. [20 points] Use the Maclauren series of $f(x) = xe^{-x}$ to estimate the value of $f(1) = \dfrac{1}{e}$ to within .01.

Solution: First we write the Maclauren series for $f(x)$. We get

$$f(x) = xe^{-x} = x\left(\sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\right) = x\left(\sum_{n=0}^{\infty}(-1)^n \frac{x^n}{n!}\right) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n!}.$$

Note the radius of convergence here is $R = \infty$ (Why is this?). Hence the value of $f(x)$ at $x = 1$ IS the value of the series evaluated at $x = 1$. We get

$$f(1) = xe^{-x}\Big|_{x=1} = \frac{1}{e} = \sum_{n=0}^{\infty}(-1)^n \frac{(1)^{n+1}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}.$$

Not knowing what this sum is, we can estimate it, knowing that it is an alternating series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \ldots$$
$$= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \ldots.$$

Recall that for an alternating series $\sum(-1)^n b_n$, that the nth partial sum s_n is closer to the actual sum than the next term b_{n+1}, or

$$|s_n - s| < b_{n+1}.$$

Seeing that the 5th term $\frac{1}{120} < \frac{1}{100} = .01$, we simply need to add the terms before this term to get an estimate for $\frac{1}{e}$ within .01 of the actual value. So

$$f(1) = \frac{1}{e} \cong 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = \frac{9}{24} = \frac{3}{8} = .375.$$

And lastly, this is not a part of this problem, but a MUCH closer approximation to $\frac{1}{e}$ is

$$\frac{1}{e} \cong 0.367879,$$

and the above estimate is off by more like .007. This problem is similar to Exercises 43 and 46, in Section 11.10 and is quite close to the example I did in class, using a power series for $\ln(1+x)$ to estimate $\ln\left(\frac{1}{2}\right)$.

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