Name: ______________________________ **Section Number:** ____________

110.109 CALCULUS II (Physical Sciences & Engineering)
SPRING 2012
MIDTERM EXAMINATION SOLUTIONS
March 7, 2012

Instructions: The exam is **7** pages long, including this title page. The number of points each problem is worth is listed after the problem number. The exam totals to one hundred points. For each item, please **show your work** or **explain** how you reached your solution. Please do all the work you wish graded on the exam. Good luck !!

PLEASE DO NOT WRITE ON THIS TABLE !!

Problem	Score	Points for the Problem
1		30
2		15
3		15
4		20
5		20
TOTAL		100

Statement of Ethics regarding this exam

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

Signature: ______________________________ **Date:** ____________

Question 1. [30 points] Do the following:

(a) Find the general antiderivative of $f(x) = \dfrac{x^2+1}{x(x^2-1)}$.

Strategy: Here, a straightforward substitution, viewing the numerator as the derivative of the denominator, will not work. Hence, we will utilize a Partial Fraction decomposition of the rational function integrand to write it as a sum of three degree-1 proper rational functions. Then we will integrate to find the anti-derivative.

Solution: Using the formula for writing this rational function as a sum of lower degree rational functions, we get

$$\frac{x^2+1}{x(x^2-1)} = \frac{x^2+1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1},$$

where each of A, B, and C are constants. Since the left-hand side must be equal to the write hand side, we can re-combine the fractions on the right-hand side to determine the values of the unknown constants. Indeed,

$$\begin{aligned}\frac{x^2+1}{x(x^2-1)} &= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} = \frac{A(x^2-1)}{x(x^2-1)} + \frac{Bx(x+1)}{x(x-1)(x+1)} + \frac{Cx(x-1)}{x(x-1)(x+1)} \\ &= \frac{Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx}{x(x^2-1)} = \frac{(A+B+C)x^2 + (B-C)x + (-A)}{x(x^2-1)}.\end{aligned}$$

Again, the denominators must be equal. Hence the numerators must be equal. Hence each coefficient on each side must be equal. This gives us three equations in three unknowns:

$$A+B+C = 1, \quad B-C = 0, \quad -A = 1.$$

This is easily solved by $A = -1$, $B = C = 1$. Hence

$$\frac{x^2+1}{x(x^2-1)} = -\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1},$$

Thus

$$\begin{aligned}\int \frac{x^2+1}{x(x^2-1)} &= \int \left(-\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1}\right) dx = -\int \frac{1}{x}\,dx + \int \frac{1}{x-1}\,dx + \int \frac{1}{x+1}\,dx \\ &= -\ln|x| + \ln|x-1| + \ln|x+1| + K,\end{aligned}$$

where K is simply the constant of integration (we already used a C above).

(b) Calculate $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-4}}$.

Strategy: Due to the quadratic polynomial under the radical in the integrand, we will use an inverse trigonometric substitution to solve for the anti-derivative.

Solution: Knowing that $\tan^2\theta = \sec^\theta - 1$, we use the substitution $x = 2\sec\theta$, $dx = 2\tan\theta\sec\theta$, and we have

$$\int \frac{dx}{x^2\sqrt{x^2-4}} = \int \frac{2\tan\theta\sec\theta}{4\sec^2\theta\sqrt{4\sec^2\theta - 4}}\,d\theta = \int \frac{2\tan\theta\sec\theta}{4\sec^2\theta|2\tan\theta|}\,d\theta.$$

Now since we have no interval on which to integrate, the absolute value signs are meaningless (we can effectively "choose" an interval where $\tan\theta$ is positive), after cancelations, we get

$$\int \frac{dx}{x^2\sqrt{x^2-4}} = \frac{1}{4}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{4}\int \cos\theta\,d\theta = \frac{1}{4}\sin\theta + C.$$

We are not done yet, since the antiderivative we are seeking is a function of x and not θ. We will need to write $\sin\theta$ in terms of x, and will employ a triangle to do this: Since $x = 2\sec\theta$, we see that $\frac{2}{x} = \cos\theta$. Thus the opposite side of a right triangle (side opposite to an angle *theta*) whose hypotenuse is x and whose adjacent side is 2 is $\sqrt{x^2-4}$. Thus $\sin\theta = \frac{\sqrt{x^2-4}}{x}$. The final answer is then

$$\int \frac{dx}{x^2\sqrt{x^2-4}} = \frac{\sqrt{x^2-4}}{4x}.$$

Check his by differentiating the last expression to recover the integrand!

Question 2. [15 points] Find the value of $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^5\theta\sec^4\theta\,d\theta$.

Strategy: We use a standard trigonometric substitution and an identity to rewrite the integrand as a polynomial in one variable. Then we anti-differentiate.

Solution: First, note that the identity $\tan^2\theta + 1 = \sec^2\theta$ will be quite useful here. Using this, and stripping off a piece of the secant part, we get

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^5\theta\sec^4\theta\,d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^5\theta\sec^2\theta\sec^2\theta\,d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^5\theta(\tan^2\theta+1)\sec^2\theta\,d\theta.$$

Now, use the substitution $u = \tan\theta$, $du = \sec^2\theta\,d\theta$, where when $\theta = -\frac{\pi}{4}$, $x = \tan\left(-\frac{\pi}{4}\right) = -1$, and when $\theta = \frac{\pi}{4}$, $x = \tan\left(\frac{\pi}{4}\right) = 1$. Then we get

$$\begin{aligned}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^5\theta\sec^4\theta\,d\theta &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^5\theta(\tan^2\theta+1)\sec^2\theta\,d\theta \\ &= \int_{-1}^{1} u^5(u^2+1)\,du = \int_{-1}^{1}(u^7+u^5)\,du = \frac{u^8}{8}+\frac{u^6}{6}\bigg|_{-1}^{1} = \frac{1}{8}+\frac{1}{6}-\frac{(-1)^8}{8}-\frac{(-1)^6}{6} = 0.\end{aligned}$$

Question 3. [15 points] Solve the following first-order Initial Value Problem:

$$x^2y' - x^5e^x = 2xy, \quad y(1) = 2.$$

Strategy: This is a linear, first order differential equation. We solve it by first calculating the integrating factor, multiplying the entire equation by the integrating factor, then integrating. We finish by solving for the unknown function.

Solution: To start, we manipulate the ODE to see if it is linear by placing it in standard form:

$$\begin{aligned} x^2y' - x^5e^x &= 2xy \\ x^2y' - 2xy &= x^5e^x \\ \frac{1}{x^2}\left[x^2y' - 2xy\right. &= \left. x^5e^x\right] \\ y' - \frac{2}{x}y &= x^3e^x. \end{aligned}$$

This gives us an integrating factor that uses the linear term $-\frac{2}{x}$ (this is our $P(x)$ in the standard form $y' + P(x)y = Q(x)$.) Thus the integrating factor is:

$$e^{\int P(x)\,dx} = e^{\int\left(-\frac{2}{x}\right)dx} = e^{-2\int\frac{1}{x}\,dx} = e^{-2\ln|x|} = e^{\ln|x|^{-2}} = x^{-2} = \frac{1}{x^2}.$$

Multiplying through, we get

$$\begin{aligned} \frac{1}{x^2}\left[y' - \frac{2}{x}y\right. &= \left. x^3e^x\right] \\ \frac{1}{x^2}y' - \frac{2}{x^3}y &= xe^x \\ \frac{d}{dx}\left[\frac{1}{x^2}y\right] &= xe^x. \end{aligned}$$

(In the last equation, we understand that the whole idea behind the integrating factor is that the resulting left-hand side is not the total derivative of a product, and we can write it that way.)

Now, integrating each side with respect to x, we get

$$\begin{aligned} \int\left(\frac{d}{dx}\left[\frac{1}{x^2}y\right]\right)dx &= \int xe^x\,dx \\ \frac{1}{x^2}y &= \int xe^x\,dx \\ &= (x-1)e^x + C. \end{aligned}$$

We can now solve for the unknown function $y(x)$, and get

$$y(x) = x^2(x-1)e^x + Cx^2.$$

Finally, using the initial data, at the point $(1, 2)$, we see that

$$2 = y(1) = (1)^2(1-1)e^1 + C(1) = C$$

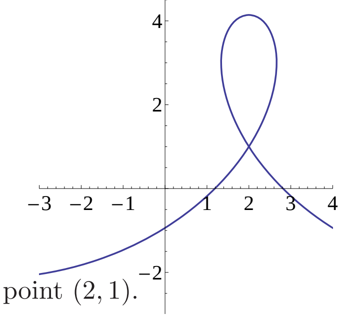
so that $C = 2$ and our particular solution to the IVP is

$$y(x) = x^2(x-1)e^x + 2x^2.$$

Question 4. [20 points] For the parameterization

$$x(t) = 2t + 2 - \pi \sin t$$
$$y(t) = 1 + \pi \cos t,$$

do the following:



(a) Find the equations of all tangent lines to the curve at the point $(2, 1)$.

Strategy: We calculate the values of the parameter t which correspond to the point $(x, y) = (2, 1)$. Then we use the parameter form of the equation for the slope of tangent lines to a parameterized curve to find the slopes of the tangent lines. The point-slope formula can then be written out explicitly for the equations of the tangent lines.

Solution: First, note that the values of t for which the curve is at the point $(2, 1)$ in the xy-plane must satisfy

$$2 = x(t) = 2t + 2 - \pi \sin t$$
$$1 = y(t) = 1 + \pi \cos t.$$

Solving the second, we get $y(t) = 1$, when $\cos t = 0$. This is true when $t = (2n + 1)\frac{\pi}{2}$, for $n \in \mathbb{Z}$ (basically an ODD integer multiple of $\frac{\pi}{2}$). However, to simultaneously solve $x(t) = 2$, we would need $0 = 2t - \pi \sin t$, or $t = \frac{\pi}{2} \sin t$. While this may be hard in general to solve, we only need to check it at the solutions we found for $y(t) = 1$. Solving, we only find two places where both equations are solved by the same values for t: $t = \pm\frac{\pi}{2}$.
The slope of the tangent line to the curve at $t = t_0$ is given in parametric form by

$$\left.\frac{dy}{dx}\right|_{(x(t_0),y(t_0))} = \frac{\left.\frac{dy}{dt}\right|_{t=t_0}}{\left.\frac{dx}{dt}\right|_{t=t_0}} = \frac{\left.\frac{d}{dt}\left[1 + \pi \cos t\right]\right|_{t=t_0}}{\left.\frac{d}{dt}\left[2t + 2 - \pi \sin t\right]\right|_{t=t_0}} = \frac{-\sin t_0}{2 - \pi \cos t_0}.$$

Evaluating at the two values of t we found that correspond to $(2, 1)$, we find

$$\left.\frac{dy}{dx}\right|_{(x(\frac{\pi}{2}),y(\frac{\pi}{2}))} = -\frac{\pi}{2}, \quad \text{and} \quad \left.\frac{dy}{dx}\right|_{(x(-\frac{\pi}{2}),y(-\frac{\pi}{2}))} = \frac{\pi}{2}.$$

Hence our two tangent lines to the parameterized curve at the point $(2, 1)$ are

$$y - 1 = -\frac{\pi}{2}(x - 2), \quad \text{and} \quad y - 1 = \frac{\pi}{2}(x - 2).$$

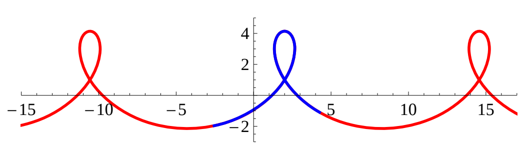
(b) Other than the point at the top of the loop, find the (x, y)-coordinates of a point on the curve where the tangent is horizontal (you cannot see it in the window for the graph provided).

Strategy: We calculate all values for t where the slope of the tangent line is zero, and evaluate the (x, y) coordinates for any of them that does NOT correspond to the top of the loop in the picture.

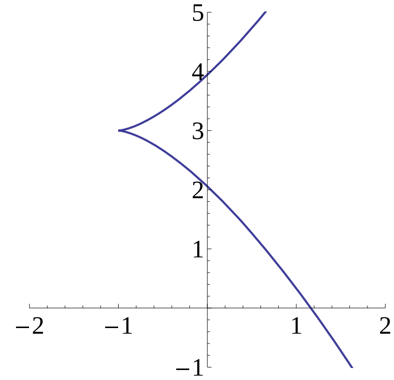
Solution: We solve the equation for t given by

$$\left.\frac{dy}{dx}\right|_{(x(t),y(t))} = \frac{-\sin t}{2 - \pi \cos t} = 0.$$

Really, we just need the condition that the numerator $-\sin t = 0$, while ensuring that the denominator $2 - \pi \cos t$ is not equal to zero at those places. We find that the numerator is zero when $t = n\pi$ for $n \in \mathbb{Z}$. And the denominator is NEVER zero at these points (it is either $2 - \pi$ or $2 + \pi$ here).
6Going back to the original equations, we find that if we choose $t = 0$, we get $x(0) = 2$, and $y(0) = 1 + \pi$. But this IS the top of the loop, so we discard this one. Choose any other integer value of n, in $t = n\pi$ and we are good. For $t = \pi$, we get $(x(\pi), y(\pi)) = (2\pi + 2, 1 - \pi)$, for example. Incidentally, here is a more complete picture of the parameterized curve. See the zero-tangent places?



Question 5. [20 points] Given the parameterization $x(t) = \frac{1}{2}t^2 - 1$ and $y(t) = 3 - \frac{1}{3}t^3$ for $t \in \mathbb{R}$, do the following:



(a) Calculate the total arc-length of the curve from $t = 0$ to $t = \sqrt{3}$.

Strategy: We use the parameterized curve version of the arc-length formula to directly calculate the quantity.

Solution: The arc-length L for a parameterized curve on the interval $t \in [a, b]$ is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Since $x'(t) = t$ and $y'(t) = t^2$, we get

$$\begin{aligned} L &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{\sqrt{3}} \sqrt{(t)^2 + (t^2)}\, dt = \int_0^{\sqrt{3}} \sqrt{t^2 + t^4}\, dt. = \int_0^{\sqrt{3}} \sqrt{t^2(t + t^2)}\, dt = \int_0^{\sqrt{3}} t\sqrt{t + t^2}\, dt. \end{aligned}$$

(The interval of integration is non-negative, hence we can pull out the t^2 from the radical as simply a positive t.) Now, with the substitution $u = 1 + t^2$, $du = 2t\,dt$, (Note: we can also use the substitution $u = \tan t$ here. It will work), where when $t = 0$, $u = 1$, and where $t = \sqrt{3}$, $u = 4$, we get

$$L = \int_0^{\sqrt{3}} t\sqrt{t + t^2}\, dt = \int_1^4 \frac{1}{2}\sqrt{u}\, du = \frac{1}{2}\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Bigg|_0^4 = \frac{1}{3}(4)^{\frac{3}{2}} - \frac{1}{3}(1)^{\frac{3}{2}} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

(b) Find the area between the curve and the x-axis from $t = 0$ to $t = 2$.

Strategy: We present two ways to find this area A: via the parameterizations, and by eliminating the parameter and recovering the xy-equation defining the curve.

Solution 1: First, note that from $t = 0$ to $t = 0$, we start at the corner in the graph and move down along the lower piece. Thus, y can be written as a function of x here, and finding the area under the curve can involve reconstructing the function $y = f(x)$ by eliminating t. Hence we start this way. Since $x = \frac{1}{2}t^2 - 1$, we can solve for t and get $t = \sqrt{2x + 2}$. Thus

$$y(t) = 3 - \frac{1}{3}t^3, \quad \text{and} \quad y(x) = 3 - \frac{1}{3}\left(\sqrt{2x + 2}\right)^3 = 3 - \frac{1}{3}(2x + 2)^{\frac{3}{2}}.$$

The area A between this curve and the x-axis is then found by standard integration by noting the limits as the x-values of the curve for $t \in [0, 2]$. These are $x(0) = -1$ and $x(2) = 1$. So

$$A = \int_{-1}^1 \left(3 - \frac{1}{3}(2x + 2)^{\frac{3}{2}}\right) dx = \int_{-1}^1 3\, dx - \frac{1}{3}\int_{-1}^1 (2x + 2)^{\frac{3}{2}}\, dx = 6 - \frac{1}{3}\int_{-1}^1 (2x + 2)^{\frac{3}{2}}\, dx.$$

We make a simple substitution (not necessary if you can "see" it) $u = 2x + 2$, $du = 2x\,dx$, and when $x = -1$, $u = 0$ and when $x = 1$, $u = 4$. We get

$$A = 6 - \frac{1}{3}\int_{-1}^1 (2x + 2)^{\frac{3}{2}}\, dx = 6 - \frac{1}{3}\left(\frac{1}{2}\right)\int_0^4 u^{\frac{3}{2}}\, du = 6 - \frac{1}{6}\left(\frac{2}{5}u^{\frac{5}{2}}\right)\Bigg|_0^4 = 6 - \frac{4^{\frac{5}{2}}}{15} = 6 - \frac{32}{15}.$$

Solution 2: The version of area under a curve when y can be written as a function of x for a parameterized curve on $t \in [a, b]$ is given by

$$A = \int_{x(a)}^{x(b)} y\, dx = \int_a^b y(t)x'(t)\, dt = \int_0^2 \left(3 - \frac{1}{3}t^3\right) t\, dt = \int_0^2 \left(3t - \frac{1}{3}t^4\right) dt.$$

We integrate directly to get

$$A = \int_0^2 \left(3t - \frac{1}{3}t^4\right) dt = \left(\frac{3}{2}t^2 - \frac{1}{15}t^5\right)\Bigg|_0^2 = \frac{3}{2}(2)^2 - \frac{1}{15}(2)^5 = 6 - \frac{32}{15}.$$

Possibly helpful formulae

- $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$
- $\sin^2\theta + \cos^2\theta = 1$
- $\sin 2\theta = 2\cos\theta\sin\theta$
- $\sin(-\theta) = -\sin\theta$
- $\sin A\cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$
- $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$
- $\tan^2\theta + 1 = \sec^2\theta$
- $\cos 2\theta = \cos^2\theta - \sin^2\theta$
- $\cos(-\theta) = \cos\theta$
- $\frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{1+x^2}$

.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	$-$