Solutions to Exam 1

1. Let \( f(x, y) = x^2 + y^2 + xy - x + y \).

(a) Show that \( x = 1, y = -1 \) is the only critical point of \( f \).

**Solution:** To find the critical point, set \( f_x = 0 \) and \( f_y = 0 \) to obtain the equations \( 2x + y - 1 = 0 \), \( 2y + x + 1 = 0 \) which has the solution \( x = 1, y = -1 \).

(b) Use the second derivative test to show that \( x = 1, y = -1 \) is a local minimum (and thus an absolute minimum it is the only critical point) of \( f \).

**Solution:** Since \( f_{xx} = 2 \) and \( D = f_{xx}f_{yy} - f_{xy}^2 = 2 \cdot 2 - 1 = 3 > 0 \), the second derivative test says that \( f \) is a local minimum.

(c) Let \( z = L(x, y) \) be the equation of the tangent plane of \( z = f(x, y) \) at the critical point. Without evaluating any integrals, explain why the following inequality holds:

\[
\int \int_R f(x, y) dA \geq \int \int_R L(x, y) dA
\]

where \( R \) is any rectangle \([a, b] \times [c, d] \).

**Solution:** The tangent plane at \( x = 1, y = -1 \) is parallel to the \( xy \)-plane (since \( f_x = 0 = f_y \) at that point) and has the equation of the form \( L(x, y) = k \) where \( k \) is a constant equal to \( f(1, -1) \). Since the local minimum \( x = 1, y = -1 \) is the only critical point of \( f \), it is an absolute minimum and hence \( f(x, y) \geq k \). This immediately implies that

\[
\int \int_R f(x, y) dA \geq \int \int_R k \, dA = \int \int_R L(x, y) dA
\]

2. Let \( g_1(x, y, z) = x^2 + y^3 + z^4 - 2 \) and \( g_2(x, y, z) = xy - y^4 + z \).

(a) Let \( h(u, v) = (u^2 + 1, v^2) \) and \( g(x, y, z) = (g_1(x, y, z), g_2(x, y, z)) \). Use the chain rule to compute the derivative of \( h \circ g \) at point \( x = 0, y = 1, z = 1 \).

**Solution:** Let \( h_1(u, v) = u^2 + 1 \) and \( h_2(u, v) = v^2 \). Then

\[
Dh = \begin{bmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{bmatrix} = \begin{bmatrix} 2u & 0 \\ 0 & 2v \end{bmatrix}
\]

Additionally,

\[
Dg = \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x & 3y^2 & 4z^3 \\ y & x - 4y^3 & 1 \end{bmatrix}
\]

At \( x = 0, y = z, z = 1 \), we have \( u = g_1(0, 1, 1) = 0, v = g_2(0, 1, 1) = 0 \), and hence,

\[
D(h \circ g)(0, 1, 1) = Dh(-2, 0)Dg(0, 1, 1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]
(b) Let the surface \( S_1 \) be given by \( g_1(x, y, z) = 0 \) and the surface \( S_2 \) by \( g_2(x, y, z) = 0 \). Let the curve \( C \) be the intersection of \( S_1 \) and \( S_2 \). Show that the point \( x = 0, y = 1, z = 1 \) is not a critical point of the function \( f(x, y, z) = 3xy^2 + y^2 + z^4 \) restricted to \( C \).

**Solution:** If \( x = 0, y = 1, z = 1 \) is a critical point, then

\[
\nabla f(0, 1, 1) = \lambda_1 \nabla g_1(0, 1, 1) + \lambda_2 \nabla g_2(0, 1, 1)
\]

(*)

for some scalars \( \lambda_1, \lambda_2 \). This leads the following system of equations:

\[
\begin{align*}
3 &= \lambda_2 \\
2 &= 3\lambda_1 - 4\lambda_2 \\
4 &= 4\lambda_1 + \lambda_2
\end{align*}
\]

The first two equations imply \( \lambda_1 = 14/3 \) and \( \lambda_2 = 3 \). But plugging this into the third equation gives \( 4 = 4(14/3) + 3 = 65/3 \), a contradiction.

(c) Give an example of a function \( f(x, y, z) \) which, when restricted to \( C \), has a critical point at \( x = 0, y = 1, z = 1 \).

**Solution:** We need a function satisfying equation (*) above for some scalars \( \lambda_1 \) and \( \lambda_2 \). For example, if we set \( \lambda_1 = \lambda_2 = 1 \), (*) gives

\[
\nabla f(0, 1, 1) = (0, 3, 4) + (1, -4, 1) = (1, -1, 5)
\]

Thus \( f(x, y, z) = x - y + 5z \) will do. (Note: There are many possible answers.)

3. Let \( f(x, y, z) = yz + xz - 6 \).

(a) At the point \( x = 1, y = 1, z = 1 \), find the unit vector that points in the direction for which \( f \) is increasing at the fastest rate.

**Solution:** The function \( f \) increases at the fastest rate in the direction of

\[
\frac{\nabla f(1, 1, 1)}{||\nabla f(1, 1, 1)||} = \frac{(1, 1, 2)}{\sqrt{6}}.
\]

(b) For \( x = (1, 1, 1) \), find a vector \( v \) for which \( \frac{d}{dt} f(x + tv) = 17 \) at \( t = 0 \). (There are many possible answers. Just find one.)

**Solution:** If we let \( v = (a, b, c) \), then

\[
17 = \frac{d}{dt} f(x + tv)|_{t=0} = \nabla f(1, 1, 1) \cdot v = (1, 1, 2) \cdot (a, b, c) = a + b + 2c.
\]

Thus, one possible answer is \( v = (17, 0, 0) \).

(c) Suppose \( c(t) \) is a flow line of \( \nabla f \) with \( c(0) = (1, 1, 1) \). Calculate the acceleration of the curve \( c(t) \) at \( t = 0 \).

**Solution:** Since \( \nabla f = (z, z, x + y) \), if we set \( c(t) = (x(t), y(t), z(t)) \), then \( c'(t) = \nabla f(c(t)) \) implies

\[
\begin{align*}
x'(t) &= z(t) \\
y'(t) &= z(t) \\
z'(t) &= x(t) + y(t)
\end{align*}
\]
Thus,
\[
\begin{align*}
x''(t) &= z'(t) = x(t) + y(t) \\
y''(t) &= z'(t) = x(t) + y(t) \\
z''(t) &= x'(t) + y'(t) = 2z(t)
\end{align*}
\]
and
\[
\begin{align*}
x''(0) &= x(0) + y(0) = 2 \\
y''(t) &= x(0) + y(0) = 2 \\
z''(t) &= 2z(0) = 2
\end{align*}
\]

4. For each of the four questions below, state whether the assertion is true or false. If it is true, justify and if it is false, give a counterexample.

(a) If \( \mathbf{a} \) and \( \mathbf{b} \) are vectors, the \( \mathbf{a} \times \mathbf{b} \) is perpendicular to \( \mathbf{b} \).

Solution: This is true. The vector triple product \( (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} \) where \( \mathbf{a} = (a_1, a_2, a_3) \) and \( \mathbf{b} = (b_1, b_2, b_3) \) is given by the determinant
\[
\begin{vmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
b_1 & b_2 & b_3
\end{vmatrix}
\]
which is equal to zero since the last two rows coincide. Two vectors whose dot product is 0 are perpendicular.

(b) If \( \mathbf{F} \) is a vector field, then \( \nabla \times \mathbf{F} \) is perpendicular to \( \mathbf{F} \).

Solution: This is false. If \( \mathbf{F}(x, y, z) = xi + zj + zk \), then \( \nabla \times \mathbf{F} = -i \). On the other hand, \( (\nabla \times \mathbf{F}) \cdot \mathbf{F} = -x \).

(c) If \( \mathbf{F}, \mathbf{G} \) and \( \mathbf{H} \) are vector fields so that \( \mathbf{F} \) is a gradient field and \( \mathbf{G} \) is a curl of some vector field, then \( \text{div } \mathbf{G} \mathbf{H} = \text{curl } \mathbf{F} \).

Solution: This is true. Since \( \mathbf{G} \) is a curl of some vector field \( \text{div } \mathbf{G} = 0 \). Since \( \mathbf{F} \) is a gradient field, \( \text{curl } \mathbf{F} = 0 \). So both the left hand side and the right hand side is equal to the zero vector.

(d) Assume \( \nabla f(x, y, z) \neq \vec{0} \) for all \( (x, y, z) \). If \( \mathbf{c}(t) \) is a flow line of \( \nabla f \), then the function \( f(\mathbf{c}(t)) \) is an increasing function of \( t \).

Solution: This is true. Using the chain rule and the fact that \( \mathbf{c}'(t) = \nabla f(\mathbf{c}(t)) \), we have
\[
\frac{d}{dt} f(\mathbf{c}(t)) = \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \mathbf{c}'(t) \cdot \mathbf{c}'(t) = |\mathbf{c}'(t)|^2 > 0
\]
and hence \( f(\mathbf{c}(t)) \) is an increasing function.