Harmonic Maps into the Model Space for the Weil-Petersson Metric

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Abstract

We investigate the behavior of harmonic maps into the metric completion of Teichmüller space with respect to the Weil-Petersson metric. In this paper, we study harmonic maps from a smooth Riemannian domain to a space modeling the normal space to a boundary strata. The main result is that the singular set of a harmonic map into this model space is of Hausdorff codimension 2. This regularity result will be used in the sequel [DMW] to study rigidity properties of Teichmüller space.

1 Introduction

Let $\mathcal{T}_{g,n}$ denote the Teichmüller space of a genus g Riemann surfaces with n punctures and $3g-3+n>0$. Endowed with the Weil-Petersson metric, $\mathcal{T}_{g,n}$ is a smooth Riemannian manifold of non-positive sectional curvature. In fact, $\mathcal{T}_{g,n}$ is a Kähler manifold whose curvature tensor is strongly negative in the sense of Siu (cf. [Siu]). The curvature condition suggests that one can apply harmonic maps theory to study $\mathcal{T}_{g,n}$. The obstacle though is that $\mathcal{T}_{g,n}$ is not metrically complete. To address this issue, one can take the metric completion $\overline{\mathcal{T}}_{g,n}$ of $\mathcal{T}_{g,n}$. Though $\overline{\mathcal{T}}_{g,n}$ is no longer a Riemannian manifold, Yamada [Ya1] observed that it is an *NPC space* (complete metric space of non-positive curvature in the sense of Alexandrov). Thus, the study of harmonic maps into $\overline{\mathcal{T}}_{g,n}$ fits in the framework of the harmonic map theory developed in [KS1] and

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[KS2]. Using this, we study rigidity problems for Teichmüller space and its mapping class group by the following program:

1. Construct an equivariant harmonic map $u : \tilde{M} \to \overline{\mathcal{T}}_{g,n}$ from a smooth Riemannian manifold into the Weil-Petersson completion of Teichmüller space.

2. Show that u maps most points of $\tilde{M}$ to $\mathcal{T}_{g,n} \subset \overline{\mathcal{T}}_{g,n}$. (More specifically, one needs to prove that u maps into $\mathcal{T}_{n,g}$ except on a set of Hausdorff dimension 2.)

3. Use a Bochner's formula to prove rigidity.

Step 1 follows from [DW]. Step 3, addressed in detail in [DMW], can be accomplished using Schumacher's calculation (cf. [Sch],[Wo]) of the complexified curvature of the Weil-Petersson metric along with Bochner formulas (cf. [Siu], [JoYa], [MSY]).

The purpose of this paper is to develop the the necessary tools to address the main step 2. More precisely, we want to understand the behavior of harmonic maps into a neighborhood the boundary $\partial\mathcal{T}_{g,n} = \overline{\mathcal{T}}_{g,n} \backslash \mathcal{T}_{g,n}$.

The boundary $\partial\mathcal{T}_{g,n}$ is stratified by lower dimensional Teichmüller spaces with each stratum being totally geodesic. Furthermore, in a neighborhood $\mathcal{N}$ of a boundary point, the Weil-Petersson metric can be approximated up to higher order by a simpler model space. Indeed, $\mathcal{N}$ is asymptotically a product $\mathcal{U} \times \mathcal{V}$ (cf. [Ya2], [DW], [Wo], [Wol2], [DM3]) where the smooth manifold $\mathcal{U}$ is an open subset of a lower dimensional Teichmüller space along with the Weil-Petersson metric and $\mathcal{V}$ is an open subset of $Y = \overline{\mathbf{H}} \times \ldots \times \overline{\mathbf{H}}$ where $\overline{\mathbf{H}}$ (referred to as the *model space*) is the metric completion of the half-plane

$$\mathbf{H} = \{(\rho, \phi) \in \mathbf{R}^2 : \rho > 0\}$$

with respect to the metric $ds^2 = 4d\rho^2 + \rho^6 d\phi^2$. The metric completion is constructed by identifying the axis $\{\rho = 0\}$ to a single point P_0 and setting

$$\overline{\mathbf{H}} = \mathbf{H} \cup \{P_0\}.$$

The major difficulty in studying harmonic maps into $\overline{\mathcal{T}}_{g,n}$ is that the target space is not locally compact. Specifically, a geodesic ball centered at a boundary point of $\overline{\mathcal{T}}_{g,n}$ is not compact. This property is

captured by the model space $\overline{\mathbf{H}}$ at the point P_0. Indeed, the geodesic ball of radius r centered at P_0,

$$B_r(P_0) = \{(\rho, \phi) \in \mathbf{R}^n : 0 < \rho < r\} \cup \{P_0\} \subset \overline{\mathbf{H}},$$

is not compact. The crucial ingredient in understanding the behavior of harmonic maps into $\overline{\mathcal{T}}_{g,n}$ is understanding the behavior of harmonic maps into $\overline{\mathbf{H}}$. The main result of this paper is the following.

Theorem 1 (Main Theorem) *If $u : \Omega \to \overline{\mathbf{H}}$ is a harmonic map from a Riemannian domain, then u maps to the interior $\mathbf{H}$ of $\overline{\mathbf{H}}$ except on a set of Hausdorff codimension 2.*

In our previous paper [DM1], we analyzed harmonic maps into

$$\overline{\mathbf{H}}_2 = \overline{\mathbf{H}}^+ \sqcup \overline{\mathbf{H}}^- / \sim, \tag{1}$$

where $\overline{\mathbf{H}}^+$ and $\overline{\mathbf{H}}^-$ denotes two distinct copies of $\overline{\mathbf{H}}$ and $\sim$ indicates that the point P_0 from each copy is identified as a single point. The key to proving Theorem 1 is to show that a harmonic map into $\overline{\mathbf{H}}$ can be closely approximated by a harmonic map into $\overline{\mathbf{H}}_2$. Indeed, we introduce of a new coordinate system (ϱ, φ) in $\mathbf{H}$ and show that the harmonic map equations of a map into $\overline{\mathbf{H}}$ with respect to the coordinates (ϱ, φ) are close to the harmonic map equation $\overline{\mathbf{H}}_2$ with respect to coordinates (ρ, ϕ) (cf. Section 4).

In the sequel [DMW], we use Theorem 1 to prove that a harmonic map $u : B_R(x_0) \to \overline{\mathcal{T}}_{g,n}$ maps to the interior $\mathcal{T}_{g,n}$ of $\overline{\mathcal{T}}_{g,n}$ except on a set of Hausdorff codimension 2. Indeed, using the coordinates on the neighborhood $\mathcal{N}$ of the boundary, we can write $u = (V, v)$ where $V : B_1(0) \to \mathbf{R}^{2j}$ and $v = (v^1, \ldots, v^m) : B_1(0) \to \overline{\mathbf{H}} \times \ldots \times \overline{\mathbf{H}}$. The maps V, $v^1, \ldots, v^m$ are not harmonic maps, but they are only *approximately harmonic* since the Weil-Petersson metric and the Weil-Petersson connection are asymptotically a product near the boundary. Applying the theory of approximately harmonic maps that we developed in [DM4] and combining it with Theorem 1, we complete the program outlined above to prove rigidity statements for Teichmüller space and its mapping class group in [DMW].

2 Preliminaries

For the sake of simplicity, we replace the metric for the model space by

$$g_{\mathbf{H}}(\rho,\phi) = \begin{pmatrix} 1 & 0 \\ 0 & \rho^6 \end{pmatrix}. \tag{2}$$

Note that $4dr^2 + r^6 d\theta^2$ is isometric $d\rho^2 + \rho^6 d\phi^2$ via the change of coordinates $\rho = 2r, \phi = \frac{\theta}{8}$. The Christoffel symbols with respect to this metric is

$$\begin{array}{ll} \Gamma^{\rho}_{\rho\rho} = 0 & \Gamma^{\phi}_{\phi\phi} = 0 \\ \Gamma^{\rho}_{\rho\phi} = 0 & \Gamma^{\phi}_{\rho\phi} = \dfrac{3}{\rho} \\ \Gamma^{\rho}_{\phi\phi} = -3\rho^5 & \Gamma^{\phi}_{\rho\rho} = 0. \end{array} \tag{3}$$

Let $d_{\mathbf{H}}$ be the distance function induced by the metric $g_{\mathbf{H}}$. The metric completion is constructed by identifying the axis $\rho = 0$ to a single point P_0 and setting

$$\overline{\mathbf{H}} = \mathbf{H} \cup \{P_0\}.$$

The distance function $d_{\mathbf{H}}$ induced by $g_{\mathbf{H}}$ is extended to $\overline{\mathbf{H}}$ by setting $d_{\mathbf{H}}(Q, P_0) = \rho$ for $Q = (\rho, \phi) \in \mathbf{H}$. The following facts are easy to check (cf. [DW]):

(1) The Riemann surface $(\mathbf{H}, g_{\mathbf{H}})$ is geodesically convex.
(2) The complete metric space $(\overline{\mathbf{H}}, d)$ is an NPC space.
(3) The space $(\overline{\mathbf{H}}, d)$ is not locally compact.

For a map $v : (\Omega, g) \to (\overline{\mathbf{H}}, d)$ from a bounded Riemannian domain, let the function $|\nabla v|^2$ be the energy density as defined in [KS1]. The *energy* of v is

$$E^v = \int_\Omega |\nabla v|^2 d\mu.$$

The map v is said to be *harmonic* if it is energy minimizing with respect to all finite energy maps with the same trace (cf. [KS1]). A harmonic map $u : (\Omega, g) \to (\overline{\mathbf{H}}, d)$ have the following important monotonicity formula. Given $x_0 \in \Omega$ and $\sigma > 0$ such that $B_\sigma(x) \subset \Omega$, identify $x_0 = 0$ via normal coordinates and let

$$E^u(\sigma) := \int_{B_\sigma(0)} |\nabla u|^2 d\mu \quad \text{and} \quad I^u(\sigma) := \int_{\partial B_\sigma(0)} d^2(u, u(x)) d\Sigma.$$

There exists a constant $c > 0$ depending only on the C^2 norm of the metric on g (with $c = 0$ when g is the standard Euclidean metric) such that

$$\sigma \mapsto e^{c\sigma^2} \frac{\sigma\ E^u(\sigma)}{I^u(\sigma)}$$

is non-decreasing. As a non-increasing limit of continuous functions,

$$Ord^u(x_0) := \lim_{\sigma \to 0} e^{c\sigma^2} \frac{\sigma\ E^u(\sigma)}{I^u(\sigma)}$$

is an upper semicontinuous function and $Ord^u(x_0) \geq 1$. The value $Ord^u(x_0)$ is called the order of u at x_0. (See Section 1.2 of [GS] with [KS1] and [KS2] justify various technical steps.) The following theorem says that most point are order 1 points.

Theorem 2 ([DM2] Theorem 16) *If $u : (\Omega, g) \to (\overline{\mathbf{H}}, d)$ is a harmonic map from a Riemannian domain, then*

$$\dim_{\mathcal{H}}(S_0(u)) \leq n - 2$$

where

$$S_0(u) = \{x \in B_1(0) : Ord^u(x) > 1\}.$$

The *homogeneous cordinates* (ρ, Φ) of $\mathbf{H}$ is defined by setting

$$\Phi = \rho^3 \phi.$$

It can be easily seen that the metric $g_{\mathbf{H}}$ is invariant under the scaling

$$\rho \to \lambda\rho,\ \ \Phi \to \lambda\Phi.$$

Thus, the distance function of $\overline{\mathbf{H}}$ is homogeneous of degree 1 under this scaling. More precisely, for P given by (ρ, Φ) in homogeneous coordinates if $P \neq P_0$ and $\lambda \in (0, \infty)$, we denote by λP the point given by

$$(\lambda\rho, \lambda\Phi) \ \text{ and } \ \lambda P_0 = P_0.$$

Then

$$d(\lambda P, \lambda Q) = \lambda d(P, Q).$$

We note that in the original coordinates (ρ, ϕ) of $\mathbf{H}$, we have

$$\lambda(\rho, \phi) = (\lambda\rho, \frac{1}{\lambda^2}\phi). \tag{4}$$

Given a harmonic map $u : (\Omega, g) \to (\overline{\mathbf{H}}, d)$, the homogeneous coordinates can be used to define blow up maps of u at $x_0 \in \Omega$. More precisely, we write

$$u = (u_\rho, u_\Phi)$$

in coordinates (ρ, Φ). After identifing $x_0 = 0$ via normal coordinates, we define

$$u_\sigma = (u_{\sigma\rho}, u_{\sigma\Phi}) : B_1(0) \to (\overline{\mathbf{H}}, d)$$

by setting

$$u_{\sigma\rho}(x) = \mu^{-1}(\sigma) u_\rho(\sigma x) \quad \text{and} \quad u_{\sigma\Phi}(x) = \mu^{-1}(\sigma) u_\Phi(\sigma x) \tag{5}$$

where

$$\mu(\sigma) = \sqrt{\frac{I^u(\sigma)}{\sigma^{n-1}}}. \tag{6}$$

The choice of the scaling constant $\mu(\sigma)$ implies that

$$I^{u_\sigma}(1) = \int_{\partial B_1(0)} d^2(u_\sigma, P_0) d\Sigma = 1. \tag{7}$$

Theorem 3 ([DM2] Corollary 15) *Let $u : (B_R(0), g) \to (\overline{\mathbf{H}}, d)$ is a harmonic map with $u(0) = P_0$ and $Ord^u(0) = 1$. There exists a sequence of blow up maps $\{u_{\sigma_i}\}$ of u at 0 converging locally uniformly in the pullback sense (cf. [KS2] Definition 3.3) to a linear function $u_* : B_1(0) \to \mathbf{R}$. In particular,*

$$d(u_{\sigma_i}(\cdot), u_{\sigma_i}(\cdot)) \to d_*(u_*(\cdot), u_*(\cdot)) \text{ uniformly on compact sets.}$$

We consider the following normalization of the blow up maps and its limit in Theorem 3. By composing with a rotation if necessary, we may assume that $\{u_{\sigma_i}\}$ converges in the pullback sense to the linear function

$$u_*(x) = Ax^1 \tag{8}$$

for some constant $A > 0$. Furthermore, let

$$c_i = \frac{u_{\sigma_i\phi}(1) + u_{\sigma_i\phi}(-1)}{2}$$

and define an isometry $T_{c_i} : \overline{\mathbf{H}} \to \overline{\mathbf{H}}$ by setting

$$T_{c_i}(P_0) = P_0 \quad \text{and} \quad T_{c_i}(\rho, \phi) = (\rho, \phi - c_i).$$

Since $(T_{c_i} \circ u_{\sigma_i})_\phi(1) = (T_{c_i} \circ u_{\sigma_i})_\phi(-1)$, we can assume that the sequence $\{u_{\sigma_i}\}$ of blow up maps in Theorem 3 satisfy

$$u_{\sigma_i \phi}(1) = u_{\sigma_i \phi}(-1), \quad \forall i = 1, 2, \ldots. \tag{9}$$

Below, we will assume that u_* and u_{σ_i} satisfy the normalization assumptions (8) and (9).

Definition 4 An arclength parameterized geodesic

$$\gamma = (\gamma_\rho, \gamma_\phi) : (-\infty, \infty) \to \mathbf{H}$$

is said to be *symmetric* if

$$\gamma_\rho(s) = \gamma_\rho(-s) \text{ and } \gamma_\phi(s) = -\gamma_\phi(-s).$$

A homogeneous degree 1 map

$$l : B_1(0) \to \mathbf{H}$$

is said to be a *symmetric homogeneous degree 1 map* if

$$l(x) = \gamma(Ax)$$

for some $A > 0$ and some symmetric geodesic. We call the number $\gamma_\phi(1)$ the *address* of a symmetric geodesic.

Lemma 5 *Let $u : (B_R(x_0), g) \to (\overline{\mathbf{H}}, d)$ be a harmonic map such that $u(x_0) = P_0$ and $Ord^u(x_0) = 1$. Furthermore, let u_* and $\{u_{\sigma_i}\}$ be as in Theorem 3 normalized such that (8) and (9) are satisfied. Then there exists a sequence of symmetric geodesics γ_{σ_i} with the address $\gamma_{\sigma_i \phi}(1) \to \infty$ and a symmetric homogeneous degree 1 maps $l_{\sigma_i} : B_1(0) \to \overline{\mathbf{H}}$ given by*

$$l_{\sigma_i}(x) = \gamma_{\sigma_i}(Ax^1)$$

such that

$$\lim_{i \to \infty} \sup_{B_r(0)} d(u_{\sigma_i}, l_{\sigma_i}) = 0$$

for any $r \in (0, R)$.

Proof. By [DM1] Lemma 8 and the normalizations (8), (9), there exists a sequence $\sigma_i \to 0$, homogeneous degree 1 maps $l'_{\sigma_i} : B_1(0) \to \overline{\mathbf{H}}$ defined by

$$l'_{\sigma_i}(x) = \begin{cases} (Ax^1, \phi_{\sigma_i}) & x^1 > 0 \\ P_0 & x^1 = 0 \\ (Ax^1, -\phi_{\sigma_i}) & x^1 < 0 \end{cases}$$

such that, for any $r \in (0,1)$,

$$\lim_{i\to\infty} \sup_{B_r(0)} d(u_{\sigma_i}, l'_{\sigma_i}) = 0.$$

We claim that $\phi_{\sigma_i} \to \infty$. Indeed, if this claim is not true, then (a subsequence of) ϕ_{σ_i} converges to ϕ_∞. Thus, l'_{σ_i} converges uniformly to a homogeneous degree 1 map

$$l'_{\sigma_i}(x) = \begin{cases} (Ax^1, \phi_\infty) & x^1 > 0 \\ P_0 & x^1 = 0 \\ (Ax^1, -\phi_\infty) & x^1 < 0 \end{cases}$$

and

$$\lim_{i\to\infty} \sup_{B_r(0)} d(u_{\sigma_i}, l'_\infty \circ R) = 0.$$

Since u_{σ_i} is a harmonic map, so is l'_∞. By the maximum principle

$$d(P_0, P) = d(l'_\infty(0), P) \neq \sup d(l'_\infty, P) = \sup_{t\in[0,1]} d((At, \pm\phi_\infty), P)$$

Taking $P = (\rho, 0)$ for ρ sufficiently large, this contradiction proves the claim. Set γ_{σ_i} to be the symmetric geodesic and with address $\gamma_{\sigma_i\rho}(1) = \phi_{\sigma_i}$. Since $\phi_{\sigma_i} \to \infty$, we have that

$$\lim_{i\to\infty} \sup_{B_1(0)} d(l_{\sigma_i}, l'_{\sigma_i}) = 0.$$

Q.E.D.

3 The foliation of H by symmetric geodesics

In the previous section (cf. Lemma 5), we showed that if a harmonic map hits the boundary P_0 of $\overline{\mathbf{H}}$, then its blow ups behave much like a symmetric geodesic. Motivated by this observation, we introduce new coordinates

$$(s,t) \text{ for } \mathbf{H} \tag{10}$$

by foliating it by a family of symmetric geodesics. More precisely, we consider

$$c = (c_\rho, c_\phi) : (-\infty, \infty) \times (-\infty, \infty) \to \mathbf{H} \tag{11}$$

satisfying the following:

- $s \mapsto c^t(s) = c(s,t)$ is a unit speed symmetric geodesic, (12)
- $c(0,1) = (1,0)$, (13)
- $s \mapsto c_\rho(s,t)$ increasing, $\lim_{s\to\pm\infty} c_\phi(s,t) = \pm\infty$, (14)
- $t \mapsto c_\rho(s,t)$ increasing, $\lim_{t\to-\infty} c_\rho(0,t) = 0, \lim_{t\to\infty} c_\rho(0,t) = \infty$, (15)
- $\left|\frac{\partial c}{\partial t}(\pm 1, t)\right| = 1, \quad \forall t \in (-\infty, \infty)$. (16)

We remark that the above conditions imply that

$$\lim_{t\to-\infty} c_\rho(\pm 1, t) = 1 \quad \text{and} \quad \lim_{t\to-\infty} c_\phi(\pm 1, t) = \infty.$$

For each $t \in (0,\infty)$, consider the vector field

$$s \mapsto X_t(s) := \frac{\partial c}{\partial t}(s,t) \quad \text{along} \quad s \mapsto c(s,t) \tag{17}$$

and set

$$J_t(s) = J(s,t) = |X_t(s)|. \tag{18}$$

Note that the symmetry of $s \mapsto c(s,t)$ implies that for all $t \in (0,\infty)$,

$$c_\phi(0,t) = 0, \quad J_t'(0) = 0 \quad \text{and} \quad < X_t(0), \frac{\partial c}{\partial s}(0,t) >= 0. \tag{19}$$

Sine it is generated by a family of geodesics, X_t is a Jacobi field and satisfies the differential equations

$$X_t''(s) + K_t(s) X_t(s) = 0 \tag{20}$$

where

$$K_t(s) = K(s,t) = -\frac{6}{c_\rho^2(s,t)} \tag{21}$$

is the Gauss curvature of $\mathbf{H}$ at $c(s,t)$. The initial conditions are

$$X_t(0) = (J_t(0), 0) \quad \text{and} \quad X_t'(0) = (0,0). \tag{22}$$

Indeed, the first initial condition above follows immediately from the orthogonality condition of (19). Furthermore, the second initial condition above follows from

$$< X_t'(s), \frac{\partial c}{\partial s} >=< \frac{\partial^2 c}{\partial s \partial t}, \frac{\partial c}{\partial s} >= \frac{1}{2}\frac{\partial}{\partial t}\left|\frac{\partial c}{\partial s}\right|^2 = 0 \tag{23}$$

and

$$\begin{aligned} < X_t'(0), \frac{\partial c}{\partial t}(0,t) > &= < \frac{\partial^2 c}{\partial s \partial t}(0,t), \frac{\partial c}{\partial t}(0,t) > \\ &= \frac{1}{2}\frac{\partial}{\partial s}\left|\frac{\partial c}{\partial t}\right|^2 (0,t) \\ &= \frac{1}{2}\frac{\partial}{\partial s} J_t^2(s)|_{s=0} = J_t(0)J_t'(0) = 0. \end{aligned}$$

For each t, X_t is orthogonal to the geodesic $s \mapsto c(s,t)$ at $s = 0$ (cf. (19)). Hence it is orthogonal for all s, i.e.

$$< X_t(s), \frac{\partial c}{\partial s}(s,t) >=< \frac{\partial c}{\partial t}, \frac{\partial c}{\partial s} > (s,t) = 0. \tag{24}$$

Equation (20) implies

$$J_t''(s) = -K(s,t)J_t(s) + J_t(s)^{-3}\left(\left|X_t'(s)\right|^2 |X_t(s)|^2 - < X_t'(s), X_t(s) >^2\right).$$

Combining (23) and (24), we conclude that both $X_t'(s)$ and $X_t(s)$ are perpendicular to $\frac{\partial c}{\partial s}$. By the two-dimensionality of $\mathbf{H}$, $X_t'(s)$ and $X_t(s)$ are parallel, and hence

$$\left|X_t'(s)\right|^2 |X_t(s)|^2 =< X_t'(s), X_t(s) >^2 .$$

The above two equalities along with (16) and (19) give us the differential equation along with boundary conditions

$$J_t'' + KJ_t = 0,\ J_t(1) = 1,\ J_t'(0) = 0. \tag{25}$$

The goal for the rest of this section is to derive estimates for $J_t(s) = \left|\frac{\partial c}{\partial t}(s,t)\right|$. These estimates are then used in the next section to derive estimates for the metric $g_{\mathbf{H}}$. We start with the following lemma.

Lemma 6 *Let $c(s,t)$ be given by (11). There exist constants $m, b \in (0,1)$ such that for $s \in [-1,1]$, we have*

$$m|s| + bc_\rho(0,t) \le c_\rho(s,t) \le |s| + c_\rho(s,0)$$

and

$$\frac{6}{(|s| + c_\rho(0,t))^2} \le -K(s,t) \le \frac{6}{(m|s| + bc_\rho(0,t))^2}.$$

Moreover, we can choose m arbitrarily close to 1 (by choosing b sufficiently close to 0).

Proof. Let $\gamma : (-\infty, \infty) \to \mathbf{H}$ be the unit speed symmetric geodesic satisfying

$$\gamma(0) = (1, 0) \text{ and } \gamma'(0) = (0, 1). \tag{26}$$

We first claim that

$$\sigma < \gamma_\rho(\sigma) \leq \sigma + 1 \quad \forall \sigma \in [0, \infty). \tag{27}$$

Indeed, for a fixed $\sigma \in [0, \infty)$, consider the function

$$f : (0, \infty) \to \mathbf{R}^+, \quad f(\rho) = d((\rho, 0), \gamma(\sigma)).$$

Since $\rho \mapsto (\rho, 0)$ is a geodesic, $f(\rho)$ is a strictly convex function. Furthermore, the geodesic γ intersects the line $\{\phi = 0\}$ perpendicularly at $(1, 0)$ by (26). Thus, $\rho = 1$ is the unique minimum of the function $f(\rho)$. In particular,

$$\sigma = d(\gamma(\sigma), (1, 0)) = f(1) < \lim_{\sigma \to 0} f(\sigma) = d(P_0, \gamma(\sigma)) = \gamma_\rho(\sigma).$$

This is the first inequality of (27). We now prove the second inequality. The second inequality is true at $\sigma = 0$ since $\gamma_\rho(0) = 1$. Now assume $\sigma \in (0, \infty)$. Note that the slope of the secant line connecting the points $(\sigma, \gamma_\rho(\sigma))$ and $(0, 1)$ on the graph the function $s \mapsto \gamma_\rho(s)$ is

$$\frac{\gamma_\rho(\sigma) - 1}{\sigma}.$$

Since $\gamma_\rho(s)$ is convex, this quantity is less than $\gamma'_\rho(\sigma) < |\gamma'(\sigma)| = 1$. This immediately implies the second inequality of (27) and ends the proof of the claim.

The inequalities of (27) implies that for a fixed $m \in (0, 1)$, we can choose $b \in (0, 1)$ sufficiently small such that

$$m\sigma + b < \gamma_\rho(\sigma) \leq \sigma + 1, \quad \forall \sigma \in [0, \infty).$$

For each $t \in (0, \infty)$, we rescale (cf. (4)) the curve $s \mapsto c^t(s) = c(s, t)$, $s \in [-1, 1]$ to define the unit speed geodesic

$$\tilde{c}^t(\sigma) : [0, c^t_\rho(0)^{-1}] \to \mathbf{H}, \quad \tilde{c}^t(\sigma) := \frac{1}{c^t_\rho(0)} c^t(c^t_\rho(0)\sigma).$$

Since $\tilde{c}^t(0) = (1, 0)$ and $\frac{\partial \tilde{c}^t}{\partial s}(0) = (0, 1)$, we conclude by (26) that $\tilde{c}^t(\sigma) = \gamma(\sigma)$. Consequently,

$$m\sigma + b \leq \tilde{c}^t_\rho(\sigma) \leq \sigma + 1, \quad \forall \sigma \in (0, c^t_\rho(0)^{-1}].$$

Multiply through by $c_\rho^t(0)$ to obtain

$$mc_\rho^t(0)\sigma + c_\rho^t(0)b \le c_\rho^t(c_\rho^t(0)\sigma) \le c_\rho^t(0)\sigma + c_\rho^t(0), \quad \forall \sigma \in (0, c_\rho^t(0)^{-1}].$$

Letting $s = c_\rho^t(0)\sigma$, we conclude

$$ms + c_\rho^t(0)b \le c_\rho^t(s) \le s + c_\rho^t(0), \quad \forall s \in [0,1].$$

By (21), we obtain

$$-6\left(ms + c_\rho^t(0)b\right)^{-2} \le K(s,t) \le -6\left(s + c_\rho^t(0)\right)^{-2}, \quad \forall s \in [0,1].$$

By symmetry, the same argument applies for $s \in [-1,0]$. Q.E.D.

Lemma 7 *For $c(s,t)$ given by (11), let*

$$a_t = c_\rho(0,t) \quad \textit{and} \quad J_t(s) = J(s,t) = \left|\frac{\partial c}{\partial t}(s,t)\right|.$$

Then there exist constants $m, b \in (0,1)$, $\beta \in (0,1)$ such that for $s \in [-1,1]$, we have

$$\left|\frac{\partial}{\partial s}\log J(s,t)\right| \le \frac{3\beta m}{m|s| + ab}$$

and

$$\frac{(m|s| + a_t b)^{3\beta}}{(m + a_t b)^{3\beta}} \le J(s,t) \le e^2 \frac{(|s| + a_t)^3}{(1 + a_t)^3}$$

Moreover, m and β can be chosen arbitrarily close to 1 by choosing b sufficiently close to 0.

Proof. Fix $t \in (0,\infty)$ and omit the subscript t for simplicity. With this notation, we rewrite (25) as

$$J'' + KJ = 0, \;\; J(1) = 1, \;\; J'(0) = 0. \tag{28}$$

Let $m, b \in (0,1)$ be as in Lemma 6 and let $\beta > 1$ be a number satisfying

$$3\beta(3\beta - 1)m^2 = 6.$$

The functions

$$j_1(s) = \frac{(s+a)^3}{(1+a)^3}$$

and

$$j_2(s) = \frac{(ms+ab)^{3\beta}}{(m+ab)^{3\beta}} \tag{29}$$

are the solutions to the differential equation

$$j_1'' + k_1 j_1 = 0, \qquad k_1 = -\frac{6}{(s+a)^2}$$

and

$$j_2'' + k_2 j_1 = 0, \qquad k_2 = -\frac{6}{(ms+ab)^2} \tag{30}$$

respectively with boundary conditions

$$j_1(1) = 1 = j_2(1).$$

For $s \in [0,1]$, Lemma 6 implies

$$-k_1 \leq -K \leq -k_2.$$

Hence

$$(J(s)j_1'(s) - J'(s)j_1(s))' \leq 0 \leq (J(s)j_2'(s) - J'(s)j_2(s))'. \tag{31}$$

Since $J'(0) = 0$ (cf. (28)), we have

$$J(0)j_1'(0) - J'(0)j_1(0) = J(0)j_1'(0)$$

and

$$J(0)j_2'(0) - J'(0)j_2(0) = J(0)j_2'(0)$$

which in turn implies by (31) that

$$J(s)j_1'(s) - J'(s)j_1(s) \leq J(0)j_1'(0) \tag{32}$$

and

$$0 \leq J(0)j_2'(0) \leq J(s)j_2'(s) - J'(s)j_2(s). \tag{33}$$

In particular,

$$0 \leq \frac{\partial}{\partial s} \log \frac{j_2(s)}{J(s)}, \quad \forall s \in [0,1]. \tag{34}$$

This implies

$$\frac{J'(s)}{J(s)} \leq \frac{j_2'(s)}{j_2(s)} = \frac{3\beta m}{ms+ab}, \quad \forall s \in [0,1]. \tag{35}$$

Furthermore, $J'' = -KJ > 0$ implies that $J'(s)$ is increasing. Since $J'(0) = 0$ (cf. (28)), we have that $J'(s) \geq 0$ for $s \in [0,1]$. This completes the proof of the first estimate of the lemma for $s \in [0,1]$. By symmetry, the same argument applies for $s \in [-1,0]$.

Integrating (34) in the interval $[s,1]$ and noting that $J(1) = j_2(1)$, we obtain

$$\frac{(ms+ab)^{3\beta}}{(m+ab)^{3\beta}} = j_2(s) \leq J(s), \quad \forall s \in [0,1]. \tag{36}$$

Since $J'(s) \geq 0$ for $s \in [0,1]$, we have that $J(0) \leq J(s)$ and

$$\begin{aligned} \frac{\partial}{\partial s}\left(\log \frac{j_1(s)}{J(s)}\right) &= \frac{j_1'(s)J(s) - J'(s)j_1(s)}{j_1(s)J(s)} \\ &\leq \frac{j_1'(0)J(0)}{j_1(s)J(s)} \quad \text{(by (32))} \\ &\leq \frac{j_1'(0)}{j_1(s)} \\ &= \frac{3a^2}{(s+a)^3} \quad \forall s \in [0,1]. \end{aligned} \tag{37}$$

Integrating this inequality in the interval $[s,1]$ and noting that $J(1) = j_1(1)$, we obtain

$$\frac{J(s)}{j_1(s)} \leq \exp\left(-\frac{3a^2}{2}\left(\frac{1}{(1+a)^2} - \frac{1}{(s+a)^2}\right)\right) \leq e^2, \quad \forall s \in [0,1]$$

and hence

$$J(s) \leq e^2 j_1(s) = e^2 \frac{(s+a)^3}{(1+a)^3}. \tag{38}$$

Combining (36) and (38), we obtain the second inequality of the lemma for $s \in [0,1]$. By symmetry, the same argument applies for $s \in [-1,0]$. Q.E.D.

Lemma 8 *For $c(s,t)$ given by (11), let*

$$a_t = c_\rho(0,t) \quad \textit{and} \quad J(s,t) = \left|\frac{\partial c}{\partial t}(s,t)\right|.$$

Then there exists $C > 0$ such that for $(s,t) \in [-1,1] \times (-\infty, 1]$, we have

$$\begin{aligned} C^{-1}(|s|+a_t)^3 \leq \quad J(s,t) &\leq C(|s|+a_t)^3. \\ \left|\frac{\partial J}{\partial s}(s,t)\right| &\leq C(|s|+a_t)^2, \\ \left|\frac{\partial^2 J}{\partial s^2}(s,t)\right| &\leq C(|s|+a_t), \\ \left|\frac{\partial^3 J}{\partial s^3}(s,t)\right| &\leq C. \end{aligned}$$

Proof. Fix $t \in (-\infty, 1]$ and omit the subscript t for simplicity. We can assume that $\frac{b}{m} < 1$ and $\beta \in (0,1)$ close to 1 in Lemma 7. Thus, for $s \in [0,1]$, we have

$$\left|\frac{\partial}{\partial s}\log J\right| \leq \frac{\frac{3\beta m}{b}}{\frac{m}{b}|s|+a} \leq \frac{\frac{3\beta m}{b}}{|s|+a}$$

and

$$\frac{b^{3\beta}\left(|s|+a\right)^{3\beta}}{2^{3\beta}} \leq \frac{b^{3\beta}\left(\frac{m}{b}|s|+a\right)^{3\beta}}{2^{3\beta}} \leq \frac{\left(m|s|+ab\right)^{3\beta}}{2^{3\beta}} \leq J \leq e^2\left(|s|+a\right)^3.$$

Combining the above, the first two estimates follow immediately. The Gauss curvature estimate of Lemma 6 states

$$0 < -K \leq \frac{\frac{6}{b^2}}{\left(\frac{m}{b}|s|+a\right)^2} \leq \frac{\frac{6}{b^2}}{\left(|s|+a\right)^2}.$$

Combining this with first inequality of the lemma and (28), we obtain implies the third estimate. Finally, differentiate (28) with respect to s to obtain

$$J''' = -K'J - KJ'.$$

Furthermore, since $c'_\rho \leq |c'| = 1$, we have

$$|K'| = \left|\frac{\partial}{\partial s}\left(\frac{-6}{c_\rho^2(s,t)}\right)\right| = \frac{12}{c_\rho^3(s,t)}\left|c'_\rho\right| \leq \frac{12}{c_\rho^3(s,t)}.$$

Lemma 6 implies

$$\frac{12}{c_\rho^3(s,t)} \leq \frac{12}{(m|s|+bc_\rho(0,t))^3} \leq \frac{\frac{12}{b^3}}{(|s|+c_\rho(0,t))^3}.$$

Combining this with the first two estimates and Lemma 6 imply the fourth estimate. Q.E.D.

Lemma 9 *For $c(s,t)$ given by (11), let*

$$a_t = c_\rho(0,t) \quad \textit{and} \quad J_t(s) = J(s,t) = \left|\frac{\partial c}{\partial t}(s,t)\right|.$$

Then there exists constant $C > 0$ such that for $(s,t) \in [-1,1] \times (-\infty, 1]$,

$$\begin{aligned} \left|\frac{\partial J}{\partial t}(s,t)\right| &\leq C(|s| + a_t)^5, \\ \left|\frac{\partial^2 J}{\partial t \partial s}(s,t)\right| &\leq C(|s| + a_t)^4, \\ \left|\frac{\partial^3 J}{\partial t \partial s^2}(s,t)\right| &\leq C(|s| + a_t)^3. \end{aligned}$$

Proof. Throughout the proof, C will denote a constant that is independent of s or t. For simplicity, we omit the subscript t and denote the t-derivative by a dot. By the assumption that $t \in (-\infty, 1]$, we can apply Lemma 8 and also to obtain the inequality $a = c_\rho(0,t) \leq 1$ (cf. (13) and (15)).

Differentiate (21) with respect to t to obtain

$$\dot{K} = \frac{\partial}{\partial t}\left(-\frac{6}{c_\rho^2}\right) = \frac{12\dot{c}_\rho}{c_\rho^3}. \tag{39}$$

By (15) and the definition of J, we have $0 \leq \dot{c}_\rho \leq J$. Thus, Lemma 6 and Lemma 8 imply

$$0 \leq \dot{K} \leq C. \tag{40}$$

Differentiate (28) with respect to t and note (16) and the symmetry of J to obtain

$$\dot{J}'' = -K\dot{J} - \dot{K}J, \quad \dot{J}(\pm 1) = 0, \ \dot{J}'(0) = 0. \tag{41}$$

If $\dot{J}$ achieves a negative local minimum at $\underline{s} \in (-\infty, \infty)$, then $0 \leq \dot{J}''(\underline{s}) = -K(\underline{s})\dot{J}(\underline{s}) - \dot{K}(\underline{s})J(\underline{s}) < 0$, a contradiction. Thus, the boundary condition $\dot{J}(\pm 1) = 0$ implies that

$$0 \leq \dot{J}(s), \quad \forall s \in [-1,1]. \tag{42}$$

By (28) and (41),

$$(J'\dot{J} - J\dot{J}')' = \dot{K}J^2.$$

Hence (40) and Lemma 8 imply that there exists a constant $C_0 > 0$ sufficiently large such that

$$(J'\dot{J} - J\dot{J}')' < 14C_0(s+a)^6 \ \text{ in } \ [-1,1]. \tag{43}$$

Furthermore, Lemma 8 and the fact that $a \leq 1$ implies there exists a constant $C_1 > 0$ sufficiently large such that

$$(s+a)^6 \leq C_1 e^{C_0(s+a)^2} J^2(s), \quad \forall s \in [-1,1]. \tag{44}$$

For constants C_0 and C_1 in (43) and (44) respectively, we claim

$$\dot{J}(s) \leq C_1 e^{C_0(s+a)^2} a^2 J(s), \quad \forall s \in [-1,1]. \tag{45}$$

Indeed, suppose (45) is not true. Then since

$$\dot{J}(\pm 1) = 0 < C_1 e^{C_0(1+a)^2} a^2 J(\pm 1),$$

there exists $s_0 \in (-1,1)$ such that

$$\dot{J}(s_0) = C_1 e^{C_0(s_0+a)} a^2 J(s_0).$$

By the symmetry of $\dot{J}$ and $\ddot{J}$, we can assume $s_0 \in (0,1]$. Let σ_0 be the smallest number in $[0, s_0)$ such that

$$C_1 e^{C_0(s+a)^2} a^2 J(s) < \dot{J}(s), \quad \forall s \in (\sigma_0, s_0). \tag{46}$$

By the definition of $\sigma_0 \in [0, s_0)$, we have the following two cases:

$$(i) \ \sigma_0 = 0 \text{ and } C_1 e^{C_0(\sigma_0+a)^2} a^2 J(\sigma_0) \leq \dot{J}(\sigma_0)$$

or

$$(ii) \ \sigma_0 \in (0, s_0) \text{ and } C_1 e^{C_0(\sigma_0+a)^2} a^2 J(\sigma_0) = \dot{J}(\sigma_0).$$

For case (i), the boundary conditions of (28) and (41) imply

$$J'(\sigma_0)\dot{J}(\sigma_0) - J(\sigma_0)\dot{J}'(\sigma_0) = J'(0)\dot{J}(0) - J(0)\dot{J}'(0) = 0. \tag{47}$$

For case (ii), the assumption that $\dot{J}(\sigma_0) = C_1 e^{-C(\sigma_0+a)^2} a^2 J(\sigma_0)$ and (46) implies that $\dot{J}(s) - C_1 e^{C(s+a)^2} a^2 J(s)$ is increasing at σ_0. Thus

$$\begin{aligned}
& C_1 e^{C_0(\sigma_0+a)^2} a^2 J'(\sigma_0) \\
\leq \ & C_1 e^{C_0(\sigma_0+a)^2} a^2 J'(\sigma_0) + 2C^2(\sigma_0+a) C_1 e^{C_0(\sigma_0+a)^2} a^2 J(\sigma_0) \\
= \ & (C_1 e^{C_0(s+a)^2} a^2 J(s))' \Big|_{s=\sigma_0} \\
\leq \ & \dot{J}'(\sigma_0).
\end{aligned}$$

Combining this inequality and equality $\dot{J}(\sigma_0) = C_1 e^{C(\sigma_0+a)^2} a^2 J(\sigma_0)$, we obtain

$$C_1 e^{C_0(\sigma_0+a)^2} a^2 J'(\sigma_0)\dot{J}(\sigma_0) \leq C_1 e^{C(\sigma_0+a)^2} a^2 J(\sigma_0)\dot{J}'(\sigma_0),$$

or more simply,

$$J'(\sigma_0)\dot{J}(\sigma_0) - J(\sigma_0)\dot{J}'(\sigma_0) \leq 0. \tag{48}$$

Integrating (43) in the interval (σ_0, s) and noting (47) for case (i) and (48) for case (ii) respectively, we obtain

$$a^2 J'(s)\dot{J}(s) - a^2 J(s)\dot{J}'(s) < 2a^2 C_0 (s+a)^7, \quad \forall s \in (\sigma_0, s_0). \tag{49}$$

Furthermore,

$$\begin{aligned} a^2(s+a)^6 &\leq C_1 e^{C_0(s+a)^2} a^2 J^2(s) && \text{(by (44))} \\ &< J(s)\dot{J}(s). \ \ \forall s \in (\sigma_0, s_0). && \text{(by (46))} \end{aligned}$$

The above two inequalities imply that for $s \in (\sigma_0, s_0)$,

$$\left(\log \frac{Ce^{C_0(s_0+a)^2} a^2 J}{\dot{J}}\right)' = \frac{a^2 J'(s)\dot{J}(s) - a^2 J(s)\dot{J}'(s)}{J(s)\dot{J}(s)} - 2C_0(s+a) < 0. \tag{50}$$

Integrating in the interval (σ_0, s_0), we obtain

$$1 = \log\left(Ce^{C_0(s_0+a)^2} \frac{a^2 J(s_0)}{\dot{J}(s_0)}\right) < \log\left(C_1 e^{C_0(\sigma_0+a)^2} \frac{a^2 J(\sigma_0)}{\dot{J}(\sigma_0)}\right) \leq 1,$$

a contradiction. This completes the proof of the claim (45). Combined with (42), we obtain

$$0 \leq \dot{J}(s) \leq Ca^2 J(s), \quad \forall s \in [-1, 1]. \tag{51}$$

Lemma 7 and (51) imply the first estimate of the lemma.

By (41), Lemma 6, the first estimate, (40) and Lemma 8,

$$|\dot{J}''(s)| = |-K\dot{J} - \dot{K}J| \leq C(s+a)^3, \quad \forall s \in [-1, 1].$$

Since $\dot{J}'(0) = 0$, integration implies

$$|\dot{J}'(s)| \leq C(s+a)^4, \quad \forall s \in [-1, 1].$$

The above two inequalities are the third and second estimates of the lemma. Q.E.D.

Lemma 10 *For $c(s,t)$ given by (11),*

$$|\dot{c}_\rho| \leq C(s+a)^3,$$

$$|\ddot{c}_\rho| \leq C(s+a)^5$$

and

$$|\check{c}_\rho| \leq C(s+a)^7$$

where $\dot{c}$, $\ddot{c}$ indicates the first and second derivatives of c with respect to t and $\check{c}$ indicates the third derivative of c with respect to t.

Proof. Throughout this proof, we define the function c'_ϕ and $\dot{c}_\phi$ by expressing c' and $\dot{c}$ as

$$\dot{c} = \dot{c}_\rho \frac{\partial}{\partial \rho} + \frac{\dot{c}_\phi}{c_\rho^3} \frac{\partial}{\partial \phi}.$$

with respect to the orthonormal basis $\{\frac{\partial}{\partial \rho}, \frac{\frac{\partial}{\partial \phi}}{c_\rho^3}\}$. Thus we obtain

$$\sqrt{(c'_\rho)^2 + (c'_\phi)^2} = |c'| = 1.$$

and

$$\sqrt{\dot{c}_\rho^2 + \dot{c}_\phi^2} = |\dot{c}| = J. \tag{52}$$

Note that with this convention, c'_ϕ and $\dot{c}_\phi$ *are not* the s- and t-derivatives of the coordinate function c_ϕ of c. In fact, there differ from these derivatives by a factor of $c_\rho^3 = \left|\frac{\partial}{\partial \phi}\right|$.

The first inequality of the lemma follows directly from (52) and Lemma 8. We now proceed with the proof of the second and the third inequality. Since $< \dot{c}, c' > \equiv 0$ by (24), we obtain

$$< \ddot{c}, c' > = \frac{\partial}{\partial t} < \dot{c}, c' > - < \dot{c}, \dot{c}' > = -\frac{\frac{\partial}{\partial s} < \dot{c}, \dot{c} >}{2} = -JJ'.$$

Additionally, (52) implies

$$< \ddot{c}, \dot{c} > = \frac{\frac{\partial}{\partial t} < \dot{c}, \dot{c} >}{2} = J\dot{J}.$$

Since

$$< c', \frac{\partial}{\partial \rho} > = c'_\rho, \qquad < \frac{\dot{c}}{J}, \frac{\partial}{\partial \rho} > = \frac{\dot{c}_\rho}{J},$$

$$< c', \frac{\partial}{\partial \phi} > = c_\rho^3 c'_\phi \qquad < \frac{\dot{c}}{J}, \frac{\partial}{\partial \phi} > = \frac{c_\rho^3 \dot{c}_\phi}{J},$$

we can express $\frac{\partial}{\partial\rho}$ and $\frac{\partial}{\partial\phi}$ in terms of the orthonormal basis $\{c', \frac{\dot{c}}{J}\}$ (cf. (24)) by

$$\frac{\partial}{\partial\rho} = c'_\rho c' + \frac{\dot{c}_\rho}{J}\frac{\dot{c}}{J} \quad \text{and} \quad \frac{\partial}{\partial\phi} = c^3_\rho c'_\phi c' + \frac{c^3_\rho \dot{c}_\phi}{J}\frac{\dot{c}}{J}.$$

Combining the above identities, we obtain

$$\begin{aligned} < \ddot{c}, \frac{\partial}{\partial\rho} > &= < \ddot{c}, c'_\rho c' + \frac{\dot{c}_\rho}{J}\frac{\dot{c}}{J} > \\ &= c'_\rho < \ddot{c}, c' > + \frac{\dot{c}_\rho}{J^2} < \ddot{c}, \dot{c} > \\ &= -c'_\rho J J' + \frac{\dot{c}_\rho \dot{J}}{J}, \end{aligned}$$

$$\begin{aligned} < \ddot{c}, \frac{\partial}{\partial\phi} > &= < \ddot{c}, c^3_\rho c'_\phi c' + \frac{c^3_\rho \dot{c}_\phi}{J}\frac{\dot{c}}{J} > \\ &= c^3_\rho c'_\phi < \ddot{c}, c' > + \frac{c^3_\rho \dot{c}_\phi}{J^2} < \ddot{c}, \dot{c} > \\ &= -c^3_\rho c'_\phi J J' + \frac{c^3_\rho \dot{c}_\phi \dot{J}}{J} \end{aligned}$$

and

$$\begin{aligned} < \dot{c}', \frac{\partial}{\partial\rho} > &= < \dot{c}', c'_\rho c' + \frac{\dot{c}_\rho}{J}\frac{\dot{c}}{J} > \\ &= c'_\rho < \dot{c}, c' > + \frac{\dot{c}_\rho}{J^2} < \dot{c}', \dot{c} > \\ &= \frac{\dot{c}_\rho}{J^2}\frac{\frac{\partial}{\partial s} < \dot{c}, \dot{c} >}{2} \\ &= \frac{\dot{c}_\rho J'}{J}. \end{aligned}$$

By (3), we obtain

$$\begin{aligned} \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial\rho} &= \dot{c}_\rho \nabla_{\frac{\partial}{\partial\rho}} \frac{\partial}{\partial\rho} + \frac{\dot{c}_\phi}{c^3_\rho} \nabla_{\frac{\partial}{\partial\phi}} \frac{\partial}{\partial\rho} \\ &= \frac{\dot{c}_\phi}{c^3_\rho} \Gamma^\phi_{\rho\phi} \frac{\partial}{\partial\phi} \\ &= \frac{3\dot{c}_\phi}{c^4_\rho} \frac{\partial}{\partial\phi} \end{aligned}$$

and

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial \phi} &= \dot{c}_\rho \nabla_{\frac{\partial}{\partial \rho}} \frac{\partial}{\partial \phi} + \frac{\dot{c}_\phi}{c_\rho^3} \nabla_{\frac{\partial}{\partial \phi}} \frac{\partial}{\partial \phi} \\
&= \dot{c}_\rho \Gamma^\phi_{\rho\phi} \frac{\partial}{\partial \phi} + \frac{\dot{c}_\phi}{c_\rho^3} \Gamma^\rho_{\phi\phi} \frac{\partial}{\partial \rho} \\
&= \frac{3\dot{c}_\rho}{c_\rho} \frac{\partial}{\partial \phi} - 3\dot{c}_\phi c_\rho^2 \frac{\partial}{\partial \rho}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\ddot{c}_\rho &= \frac{\partial}{\partial t} < \dot{c}, \frac{\partial}{\partial \rho} > \\
&= < \ddot{c}, \frac{\partial}{\partial \rho} > + < \dot{c}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial \rho} > \\
&= -c'_\rho J J' + \frac{\dot{c}_\rho \dot{J}}{J} + \frac{3\dot{c}_\phi^2}{c_\rho}, \qquad (53)
\end{aligned}$$

$$\begin{aligned}
\ddot{c}_\phi &= \frac{\partial}{\partial t} < \dot{c}, \frac{\partial}{\partial \phi} > \\
&= < \ddot{c}, \frac{\partial}{\partial \phi} > + < \dot{c}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial \phi} > \\
&= -c_\rho^3 c'_\phi J J' + \frac{c_\rho^3 \dot{c}_\phi \dot{J}}{J} + 3c_\rho^2 \dot{c}_\rho \dot{c}_\phi - 3\dot{c}_\rho \dot{c}_\phi c_\rho^2 \qquad (54)
\end{aligned}$$

and

$$\begin{aligned}
\dot{c}'_\rho &= \frac{\partial}{\partial t} < c', \frac{\partial}{\partial \rho} > \\
&= < \dot{c}', \frac{\partial}{\partial \rho} > + < c', \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial \rho} > \\
&= \frac{\dot{c}_\rho J'}{J} + \frac{3(\dot{c}_\phi)^2}{c_\rho}. \qquad (55)
\end{aligned}$$

Thus, applying Lemma 8 and Lemma 9 to the right hand side of (53), we obtain the first inequality of the lemma. Differentiating (53) with respect to t and applying (54) and (55), we obtain the second estimate. Q.E.D.

Lemma 11 *For $c(s,t)$ given by (11), let*

$$a_t = c_\rho(0,t) \quad \textit{and} \quad J_t(s) = J(s,t) = \left|\frac{\partial c}{\partial t}(s,t)\right|.$$

Then there exists constant $C > 0$ such that for $(s,t) \in [-1,1] \times (-\infty, 1]$,

$$\begin{aligned} \left|\frac{\partial^2 J}{\partial t^2}(s,t)\right| &\leq C(|s| + a_t)^7, \\ \left|\frac{\partial^3 J}{\partial s \partial t^2}(s,t)\right| &\leq C(|s| + a_t)^6, \\ \left|\frac{\partial^4 J}{\partial s^2 \partial t^2}(s,t)\right| &\leq C(|s| + a_t)^5. \end{aligned}$$

Proof. Differentiate (41) and note (16) to obtain

$$\ddot{J}'' = -K\ddot{J} - \ddot{K}J - 2\dot{K}\dot{J}, \quad \ddot{J}'(0) = 0, \quad \ddot{J}(1) = 0. \tag{56}$$

Differentiate (39) to obtain

$$\ddot{K} = -\frac{36\dot{c}_\rho^2}{c_\rho^4} + \frac{12\ddot{c}_\rho}{c_\rho^3}. \tag{57}$$

Applying Lemma 10, we thus obtain

$$\left|\ddot{K}\right| \leq C(s+a)^2. \tag{58}$$

Thus, by (58), Lemma 8, (40) and Lemma 9 imply that there exists $C_0 > 0$ such that

$$|\ddot{K}J + 2\dot{K}\dot{J}| \leq C_0(s+a)^5 \quad \text{in} \quad [-1,1]. \tag{59}$$

By Lemma 6, there exists a constant $C_1 > 0$ such that

$$-K \leq \frac{C_1}{(s+a)^2} \tag{60}$$

With C_0 as in (59) and C_1 as in (60), suppose that there exists $s \in (-1,1)$ such that $\ddot{J}(s) < -\frac{C_0}{C_1}(s+a)^7$. Since $\ddot{J}(\pm 1) = 0$, this implies that $\ddot{J}$ achieves local minimum $\ddot{J}(\underline{s}) < -\frac{C_0}{C_1}(s+a)^7$ at some point $\underline{s} \in (-1,1)$. Thus, (60) implies $-K(\underline{s})\ddot{J}(\underline{s}) < -C_0(\underline{s}+a)^5$. By (56) and (59), we conclude $0 \leq \ddot{J}''(\underline{s}) < 0$. This contradiction proves

$$-C(s+a)^7 \leq \ddot{J}, \quad \forall s \in [-1,1]. \tag{61}$$

By (28), (56), (59) and Lemma 8,

$$(J'\ddot{J} - J\ddot{J}')' = (\ddot{K}J + 2\dot{K}\dot{J})J \leq C(s+a)^8. \tag{62}$$

By comparing (62) to (43), we observe that by following the proof of (45), we obtain

$$\ddot{J}(s) \leq C_1 e^{-C_0(s+a)^2} a^4 J, \quad \forall s \in [-1,1]. \tag{63}$$

Combined with (61), we obtain

$$-(s+a)^7 \leq \ddot{J} \leq Ca^4 J.$$

Thus, Lemma 7 and (51) imply the first estimate of the lemma. By (56), Lemma 6, the first estimate and (59),

$$|\ddot{J}''(s)| = |-K\ddot{J} - \ddot{K}J - 2\dot{K}\dot{J}| \leq C(s+a)^3, \quad \forall s \in [-1,1].$$

Since $\dot{J}'(0) = 0$, integration implies

$$|\ddot{J}'(s)| \leq C(s+a)^4, \quad \forall s \in [-1,1].$$

The above two inequalities are the third and second estimates of the lemma. Q.E.D.

Lemma 12 *For $c(s,t)$ given by (11), let*

$$a_t = c_\rho(0,t) \quad \textit{and} \quad J_t(s) = J(s,t) = \left|\frac{\partial c}{\partial t}(s,t)\right|.$$

Then there exists constant $C > 0$ such that for $(s,t) \in [-1,1] \times (-\infty, 1]$,

$$\begin{aligned}
\left|\frac{\partial^3 J}{\partial t^3}(s,t)\right| &\leq C(|s| + a_t)^9, \\
\left|\frac{\partial^4 J}{\partial s \partial t^3}(s,t)\right| &\leq C(|s| + a_t)^8, \\
\left|\frac{\partial^5 J}{\partial s^2 \partial t^3}(s,t)\right| &\leq C(|s| + a_t)^7.
\end{aligned}$$

Proof. Differentiate (56) and note (16) to obtain (with the third derivative of J and K with respect to t denoted by $\check{J}$ and $\check{K}$ respectively)

$$\check{J}'' = -K\check{J} - \check{K}J - 2\ddot{K}\dot{J} - 2\dot{K}\ddot{J}, \quad \ddot{J}'(0) = 0, \quad \ddot{J}(1) = 0. \tag{64}$$

Differentiate (57) to obtain

$$\check{K} = \frac{144\dot{c}_\rho^3}{c_\rho^5} - \frac{144\dot{c}_\rho\ddot{c}_\rho}{c_\rho^4} + \frac{12\check{c}_\rho}{c_\rho^3}. \tag{65}$$

Applying Lemma 10, we thus obtain

$$|\check{K}| \leq C(s+a)^4 \quad \text{in} \quad [-1,1]. \tag{66}$$

Thus, (66), Lemma 8, (58), Lemma 9, (40) and Lemma 11 imply that there exists $C_0 > 0$ such that

$$|\check{K}J + 2\ddot{K}\dot{J} + 2\dot{K}\ddot{J}| \leq C_0 \leq (s+a)^7 \quad \text{in} \quad [-1,1]. \tag{67}$$

Next, combining (28) and (64), we obtain

$$(J'\check{J} - \check{J}'J)' = \left(\check{K}J + 2\ddot{K}\dot{J} + 2\dot{K}\ddot{J}\right)J \leq C(s+a)^{10} \quad \text{in} \quad [-1,1]. \tag{68}$$

By comparing (67) to (59) and (68) to (62), we observe that the follows from the proof of Lemma 11 q.e.d.

4 Metric estimates of the model space

In this section, we use the family of symmetric geodesics $c(s,t)$ given in the previous section to define a new coordinate system for $\mathbf{H}$. We then give estimates of the metric $g_{\mathbf{H}}$ and Christoffel symbols represented in terms these new coordinates. The motivation of using $c(s,t)$ is that blow ups of a harmonic map are closely approximated by symmetric homogeneous degree 1 maps (cf. Lemma 5). With the estimates derived in this section, we derive estimates of harmonic maps in Section 5 below.

Given a symmetric homogeneous degree 1 map $l : B_1(0) \to \mathbf{H}$ defined in Definition 4, let

$$l(x) = (l_s(x), l_t(x))$$

be the expression of l with respect to the coordinates (s,t). Then by the construction of the coordinates (s,t),

$$\exists t_* > 0 \text{ such that } l_t(x) \equiv t_*. \tag{69}$$

For simplicity, apply a linear change of variables

$$(s,t) \mapsto (\varrho, \varphi) = (s, t - t_*) \tag{70}$$

such that in coordinates (ϱ, φ),

$$l(x) = (l_\varrho(x), l_\varphi(x)) = (Ax^1, 0).$$

We goal is to compare the geometry of our space $\mathbf{H}$ to the space $\overline{\mathbf{H}}_2$ studied in [DM1]. Recall that $\overline{\mathbf{H}}_2$ is constructed by taking two copies $\overline{\mathbf{H}}^+$ and $\overline{\mathbf{H}}^-$ of $\overline{\mathbf{H}}$ and identifying the point P_0 (cf. (1)). Consider coordinates on $\overline{\mathbf{H}}_2 \backslash \{P_0\}$ by first applying the change of variables $(\rho, \phi) \mapsto (-\rho, \phi)$ to obtain new coordinates for $\overline{\mathbf{H}}^-$. Thus, we then have coordinates

$$(\rho, \phi) \in \mathbf{R} \backslash \{0\} \times \mathbf{R} \tag{71}$$

for $\overline{\mathbf{H}}_2 \backslash \{P_0\}$ with the property that $\rho > 0$ implies $(\rho, \phi) \in \mathbf{H}^+$ and $\rho < 0$ implies $(\rho, \phi) \in \mathbf{H}^-$. The metric and the Christoffel symbols of $\overline{\mathbf{H}}_2 \backslash \{P_0\}$ expressed in the coordinates (ρ, ϕ) are

$$\begin{pmatrix} 1 & 0 \\ 0 & \rho^6 \end{pmatrix} \tag{72}$$

and

$$\begin{array}{ll} \Gamma^\rho_{\rho\rho} = 0 & \Gamma^\phi_{\phi\phi} = 0 \\ \Gamma^\rho_{\rho\phi} = 0 & \Gamma^\phi_{\rho\phi} = \dfrac{3}{\rho} \\ \Gamma^\rho_{\phi\phi} = -3\rho^5 & \Gamma^\phi_{\rho\rho} = 0. \end{array} \tag{73}$$

We want to compare the above metric expression and the Christoffel symbols to that of of $g_{\mathbf{H}}$ in terms of our new coordinates given in (70). By construction (cf. (12), (18) and (24)), the metric $g_{\mathbf{H}}$ with respect to coordinates (ϱ, φ) is

$$g_{\mathbf{H}}(\varrho, \varphi) = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{J}^2(\varrho, \varphi) \end{pmatrix}$$

where we set

$$\mathcal{J}(\varrho, \varphi) := \left| \frac{\partial c}{\partial t}(\varrho, \varphi + t_*) \right| = J(\varphi, \varrho + t_*).$$

We will assume $(\varrho, \varphi) \in [-1,1] \times (-\infty, 1-t_*]$ in order to apply Lemma 8, Lemma 9, Lemma 11 and Lemma 12. For simplicity, introduce the function

$$\wp = \wp(\varrho, \varphi)$$

given by

$$\wp = |\varrho| + c_\rho(0, \varphi + t_*) = |s| + a_t. \tag{74}$$

Here one should observe that in $\overline{\mathbf{H}}_2$, $|\rho|$ is the distance of a point (ρ, ϕ) to the singular point P_0. Recall that P_0 is the point defined by identifying the entire line $\{\rho = 0\}$ as a single point. In coordinates (ϱ, φ) of $\mathbf{H}$, the line $\{\varrho = 0\}$ plays the role of the singular point P_0. Indeed, $\{\varrho = 0\}$ has the geometric property that the family of geodesic lines $\{\varphi = const.\}$, is squeezed together at $\varrho = 0$ as $const. \to -\infty$. The distance of the point (ϱ, φ) to $\{\varrho = 0\}$ is $|\varrho|$. Thus, $\wp$ is the distance to $\{\varrho = 0\}$ modulo the error of a_t. Note that by (15), we have that $a_t = c_\rho(0, t) \to 0$ as $t \to -\infty$. Therefore, we observe that for a fixed φ,

$$\lim_{t_* \to -\infty} c_\rho(0, \varphi + t_*) = 0.$$

To summarize, the line $\{\varrho = 0\}$ is asymptotically the singular point P_0 in $\overline{\mathbf{H}}_2$ and the parameter $\wp$ is asymptotically the parameter ρ in $\overline{\mathbf{H}}$, i.e. distance to the line $\{\varrho = 0\}$.

In [DM1], we considered the subset of $\overline{\mathbf{H}}_2$ consisting of geodesic lines; more specifically, in terms of the coordinates (ρ, ϕ) and a fixed $\phi_0 > 0$, we define the subset

$$\overline{\mathbf{H}}_2[\phi_0] := \{(\rho, \phi) \in \overline{\mathbf{H}}_2 : |\phi| \leq \phi_0\}.$$

Note that $\overline{\mathbf{H}}_2[\phi_0]$ is totally geodesic since the boundary lines $\phi = \phi_0$ and $\phi = -\phi_0$ are images of geodesic lines. Define an analogous totally geodesic subset of $\overline{\mathbf{H}}$; more specifically, in terms of the coordinates (ϱ, φ) and a fixed $\varphi_0 > 0$, define the subset

$$\overline{\mathbf{H}}[\varphi_0] := \{(\varrho, \varphi) \in \overline{\mathbf{H}} : |\varphi| \leq \varphi_0\}.$$

Implicit in this definition is that $\overline{\mathbf{H}}[\varphi_0]$ depends on the choice of t_* since the coordinates (ϱ, φ) depends on t_* (cf. (70)). The estimates derived below for the metric $g_{\mathbf{H}}$ and Christoffel symbols expressed with respect to the coordinates (ϱ, φ) show that $\overline{\mathbf{H}}[\varphi_0]$ geometrically approximates $\overline{\mathbf{H}}_2[\phi_0]$. We shall make this statement more precise (in terms of harmonic maps) in Section 5.

By Lemma 8, there exists a constant $C > 0$ such that (compare this to (72))

$$C^{-1}\begin{pmatrix} 1 & 0 \\ 0 & \wp^6 \end{pmatrix} \leq \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{J}^2(\varrho,\varphi) \end{pmatrix} \leq C\begin{pmatrix} 1 & 0 \\ 0 & \wp^6 \end{pmatrix}. \tag{75}$$

Furthermore, by Lemma 8, Lemma 9, Lemma 11, Lemma 12 and the fact that $(s,t) \mapsto (\varrho,\varphi)$ is a linear change of variables (and thus $\frac{\partial}{\partial s} = \frac{\partial}{\partial \varrho}$, $\frac{\partial}{\partial t} = \frac{\partial}{\partial \varphi}$) yield the *derivative estimates of* $\mathcal{J}$; i.e. there exists $C > 0$ such that

$$\begin{aligned}
&\left|\frac{\partial \mathcal{J}}{\partial \varrho}(\varrho,\varphi)\right| \leq C\wp^2, && \left|\frac{\partial^2 \mathcal{J}}{\partial \varrho^2}(\varrho,\varphi)\right| \leq C\wp, && \left|\frac{\partial^3 \mathcal{J}}{\partial \varrho^3}(\varrho,\varphi)\right| \leq C. \\
&\left|\frac{\partial \mathcal{J}}{\partial \varphi}(\varrho,\varphi)\right| \leq C\wp^5, && \left|\frac{\partial^2 \mathcal{J}}{\partial \varphi \partial \varrho}(\varrho,\varphi)\right| \leq C\wp^4, && \left|\frac{\partial^3 \mathcal{J}}{\partial \varphi \partial \varrho^2}(\varrho,\varphi)\right| \leq C\wp^3 \\
&\left|\frac{\partial^2 \mathcal{J}}{\partial \varphi^2}(\varrho,\varphi)\right| \leq C\wp^7, && \left|\frac{\partial^3 \mathcal{J}}{\partial \varphi^2 \partial \varrho}(\varrho,\varphi)\right| \leq C\wp^6, && \left|\frac{\partial^4 \mathcal{J}}{\partial \varphi^2 \partial \varrho^2}(\varrho,\varphi)\right| \leq C\wp^5 \\
&\left|\frac{\partial^3 \mathcal{J}}{\partial \varphi^3}(\varrho,\varphi)\right| \leq C\wp^{10}, && \left|\frac{\partial^4 \mathcal{J}}{\partial \varphi^3 \partial \varrho}(\varrho,\varphi)\right| \leq C\wp^9, && \left|\frac{\partial^5 \mathcal{J}}{\partial \varphi^3 \partial \varrho^2}(\varrho,\varphi)\right| \leq C\wp^8.
\end{aligned} \tag{76}$$

By standard computation, Christoffel symbols with respect to the coordinates (ϱ,φ) are

$$\begin{aligned}
&\Gamma^{\varrho}_{\varrho\varrho} = 0 && \Gamma^{\varphi}_{\varphi\varphi} = \frac{1}{2}\frac{\partial}{\partial \varphi}\log \mathcal{J}^2 = \frac{\partial}{\partial \varphi}\log \mathcal{J} \\
&\Gamma^{\varrho}_{\varrho\varphi} = 0 && \Gamma^{\varphi}_{\varrho\varphi} = \frac{1}{2}\frac{\partial}{\partial \varrho}\log \mathcal{J}^2 = \frac{\partial}{\partial \varrho}\log \mathcal{J} \\
&\Gamma^{\varrho}_{\varphi\varphi} = -\frac{1}{2}\frac{\partial}{\partial \varrho}\mathcal{J}^2 = -\left(\frac{\partial \mathcal{J}}{\partial \varrho}\right)\mathcal{J} && \Gamma^{\varphi}_{\varrho\varrho} = 0.
\end{aligned} \tag{77}$$

Thus, we can apply (76) to obtain the following *Christoffel symbols estimates*: there exists $C > 0$ such that (compare this to (73) and its derivatives)

$$\begin{aligned}
\left|\Gamma^{\varrho}_{\varphi\varphi}\right|(\varrho,\varphi) &\leq C\wp^5 \\
\left|\Gamma^{\varphi}_{\varrho\varphi}\right|(\varrho,\varphi) &\leq \frac{C}{\wp} \\
\left|\Gamma^{\varphi}_{\varphi\varphi}\right|(\varrho,\varphi) &\leq C\wp^2 \\
\left|\frac{\partial}{\partial \varrho}\Gamma^{\varrho}_{\varphi\varphi}\right|(\varrho,\varphi) &\leq C\wp^4
\end{aligned}$$

$$
\begin{aligned}
\left|\frac{\partial^2}{\partial\varrho^2}\Gamma^{\varrho}_{\varphi\varphi}\right|(\varrho,\varphi) &\leq C\wp^3 \\
\left|\frac{\partial}{\partial\varphi}\Gamma^{\varrho}_{\varphi\varphi}\right|(\varrho,\varphi) &\leq C\wp^7 \\
\left|\frac{\partial^2}{\partial\varphi^2}\Gamma^{\varrho}_{\varphi\varphi}\right|(\varrho,\varphi) &\leq C\wp^6 \\
\left|\frac{\partial^2}{\partial\varphi\partial\varrho}\Gamma^{\varrho}_{\varphi\varphi}\right|(\varrho,\varphi) &\leq C\wp^5. \qquad (78)
\end{aligned}
$$

5 Estimates for harmonic maps

In [GrSc], Gromov and Schoen introduced the notion of a totally geodesic subset being *essentially regular*. Loosely speaking, this means a harmonic map into this subset can be well-approximated by homogeneous degree 1 maps. In this section, we show that $\mathbf{H}$ resembles $\overline{\mathbf{H}}_2$ in a crucial way: the totally geodesic subspace $\overline{\mathbf{H}}[\varphi_0]$ of $\mathbf{H}$ (introduced in Section 5) and its counterpart $\overline{\mathbf{H}}_2[\phi_0]$ (introduced in [DM1]) of $\overline{\mathbf{H}}_2$ satisfy a property which is similar to being essentially regular. More precisely, we prove that harmonic maps into $\overline{\mathbf{H}}[\varphi_0]$ are regular in a sense given by Theorem 16 below. We remark that so far we are unable to prove that $\overline{\mathbf{H}}_2[\phi_0]$ or $\overline{\mathbf{H}}[\varphi_0]$ is essentially regular in the strict sense of [GrSc]. However, the weaker notion that we prove in Theorem 16 is sufficient for our applications.

For a harmonic map $w : (B_1(0), g) \to (\mathbf{H}, d_{\mathbf{H}})$, write

$$w = (w_\varrho, w_\varphi)$$

in our new coordinates (ϱ, φ) of $\overline{\mathbf{H}}$. For simplicity, set $\wp_w$ to be the composition of $\wp$ defined in (74) and the map w; i.e.

$$\wp_w := \wp(w_\varrho, w_\varphi) = |w_\rho| + c(0, w_\varphi + t_*). \qquad (79)$$

Fix $R \in [\frac{3}{4}, 1)$. The map w is locally Lipschitz continuous (cf. [KS1] Theorem 2.4.6); in particular,

$$|\nabla w| = \left(|\nabla w_\varrho|^2 + |\mathcal{J}(w_\rho, w_\varrho)|^2 \, |\nabla w_\varphi|^2\right)^{\frac{1}{2}} \leq L \text{ in } B_R(0)$$

for some constant L that depends on the dimension n of the domain, R and the total energy of w. Thus, (75) and (79) imply

$$|\nabla w_\varrho| \leq L \quad \text{and} \quad \wp_w^3 |\nabla w_\varphi| \leq L \quad \text{in} \quad B_R(0). \qquad (80)$$

The harmonic map equations are (cf. (77))

$$\triangle w_\varrho = -\Gamma^\varrho_{\varphi\varphi}(w_\varrho, w_\varphi)|\nabla w_\varphi|^2 \tag{81}$$

and

$$\begin{aligned}
\triangle w_\varphi &= -\Big(2\Gamma^\varphi_{\varrho\varphi}(w_\varrho, w_\varphi)\nabla w_\varphi \cdot \nabla w_\varrho + \Gamma^\varphi_{\varphi\varphi}(w_\varrho, w_\varphi)|\nabla w_\varphi|^2\Big) \\
&= -\frac{\Big(2\frac{\partial \mathcal{J}}{\partial \varrho}(w_\varrho, w_\varphi)\nabla w_\varphi \cdot \nabla w_\varrho + \frac{\partial \mathcal{J}}{\partial \varphi}(w_\varrho, w_\varphi)|\nabla w_\varphi|^2\Big)}{\mathcal{J}(w_\varrho, w_\varphi)} \\
&= -\frac{\Big(2\nabla \mathcal{J}(w_\varrho, w_\varphi) \cdot \nabla w_\varphi - \frac{\partial \mathcal{J}}{\partial \varphi}(w_\varrho, w_\varphi)|\nabla w_\varphi|^2\Big)}{\mathcal{J}(w_\varrho, w_\varphi)}. \qquad (82)
\end{aligned}$$

Thus, the derivative estimate of the metric (76), the Christoffel symbols estimates (78) and Lipschitz estimates (80) imply

$$\begin{aligned}
|\triangle w_\varrho| &\le C\wp_w^5|\nabla w_\varphi|^2 \le \frac{CL^2}{\wp_w} \\
|\triangle w_\varphi| &\le C\Big(\wp_w^{-1}|\nabla w_\varrho||\nabla w_\varphi| + \wp_w^2|\nabla w_\varphi|^2\Big) \le \frac{CL^2}{\wp_w^4}. \qquad (83)
\end{aligned}$$

in $B_R(0)$. Define

$$w_\Upsilon := w_\varrho + \frac{w_\varphi^2}{2}\Gamma^\varrho_{\varphi\varphi}(w_\varrho, w_\varphi) \ \text{ and } \ w_\Phi := \mathcal{J}(w_\varrho, w_\varphi)w_\varphi - \frac{1}{2}\frac{\partial \mathcal{J}}{\partial \varphi}(w_\varrho, w_\varphi)w_\varphi^2.$$

We are now in position to prove the main result of this section (Lemma 14 below). We first need the following definition.

Definition 13 A smooth Riemannian metric g on $B_R(0) \subset \mathbf{R}^n$ is said to be *normalized* if the standard Euclidean coordinates $(x^1, \ldots, x^n)$ are normal coordinates of g. The metric g_s for $s \in (0, R]$ on $B_1(0)$ is defined by

$$g_s(x) = g(sx).$$

Lemma 14 *Let $R \in [\frac{1}{2}, 1)$, $E_0 > 0$, $A_0 > 0$ and a normalized metric g on $B_R(0)$ be given. Then there exists $C_0 > 0$ such that if $\phi_0 > 0$, $s \in (0, 1]$ and $w : (B_R(0), g_s) \to \overline{\mathbf{H}}[\varphi_0]$ is a harmonic map with*

$$E^w \le E_0 \ \text{ and } \ |w_\varrho|_{L^\infty(B_{\frac{15R}{16}}(0))} \le A_0,$$

then

$$|\mathcal{J}(w_\varrho, w_\varphi)\nabla w_\varphi|_{L^\infty(B_{\frac{15R}{16}}(0))} \le C_0\phi_0,$$

and

$$|\triangle w_\Upsilon|_{L^\infty(B_{\frac{15R}{16}}(0))} \leq C_0\phi_0^2.$$

Proof. Through out the proof, C_0 will denote a generic constant dependent ony on E_0, A_0 and g. The assumption on the bounds of $|w_\rho|$ and $|w_\phi|$ imply that

$$|w_\Phi|_{L^\infty(B_R(0))} = |w_\rho^3 w_\phi|_{L^\infty(B_R(0))} \leq C\phi_0. \tag{84}$$

Since the harmonic map equation (82) implies

$$2\nabla\mathcal{J}(w_\varrho, w_\varphi)\cdot\nabla w_\varphi + \mathcal{J}(w_\varrho, w_\varphi)\triangle w_\varphi - \frac{\partial\mathcal{J}}{\partial\varphi}(w_\varrho, w_\varphi)|\nabla w_\varphi|^2 = 0,$$

we have

$$\begin{aligned}
\triangle w_\Phi &= \triangle\left(\mathcal{J}(w_\varrho, w_\varphi)w_\varphi - \frac{1}{2}\frac{\partial\mathcal{J}}{\partial\varphi}(w_\varrho, w_\varphi)w_\varphi^2\right) \qquad (85)\\
&= \triangle\mathcal{J}(w_\varrho, w_\varphi)w_\varphi + 2\nabla\mathcal{J}(w_\varrho, w_\varphi)\cdot\nabla w_\varphi + \mathcal{J}(w_\varrho, w_\varphi)\triangle w_\varphi\\
&\quad -\frac{1}{2}\triangle\frac{\partial\mathcal{J}}{\partial\varphi}(w_\varrho, w_\varphi)w_\varphi^2 - \nabla\frac{\partial\mathcal{J}}{\partial\varphi}(w_\varrho, w_\varphi)\cdot\nabla w_\varphi w_\varphi\\
&\quad -\frac{\partial\mathcal{J}}{\partial\varphi}(w_\varrho, w_\varphi)\triangle w_\varphi w_\varphi - \frac{\partial\mathcal{J}}{\partial\varphi}(w_\varrho, w_\varphi)|\nabla w_\varphi|^2\\
&= \triangle\mathcal{J}(w_\varrho, w_\varphi)w_\varphi\\
&\quad -\frac{1}{2}\triangle\frac{\partial\mathcal{J}}{\partial\varphi}(w_\varrho, w_\varphi)w_\varphi^2 - \nabla\frac{\partial\mathcal{J}}{\partial\varphi}(w_\varrho, w_\varphi)\cdot\nabla w_\varphi w_\varphi\\
&\quad -\frac{\partial\mathcal{J}}{\partial\varphi}(w_\varrho, w_\varphi)\triangle w_\varphi w_\varphi.
\end{aligned}$$

We now estimates the terms on the right hand side of (85). First,

$$\begin{aligned}
\triangle\mathcal{J}(w_\varrho, w_\varphi) &= \frac{\partial\mathcal{J}}{\partial\varrho}\triangle w_\varrho + \frac{\partial^2\mathcal{J}}{\partial\varrho^2}|\nabla w_\varrho|^2 + \frac{\partial\mathcal{J}}{\partial\varphi}\triangle w_\varphi + \frac{\partial^2\mathcal{J}}{\partial\varphi^2}|\nabla w_\varphi|^2\\
&\quad +2\frac{\partial^2\mathcal{J}}{\partial\varphi\partial\varrho}\nabla w_\varrho\cdot\nabla w_\varphi. \qquad (86)
\end{aligned}$$

By the metric derivative estimates (76), the Lipschitz estimate (80) and the bounds of the Laplacians (83), the five terms on the right hand side has a bound in $B_R(0)$ of

$$\left|\frac{\partial\mathcal{J}}{\partial\varrho}\triangle w_\varrho\right| = \left|C\wp_w^2\frac{CL^2}{\wp_w}\right|$$

$$
\begin{aligned}
& \leq CL^2\wp_w \\
\left|\frac{\partial^2\mathcal{J}}{\partial\varrho^2}|\nabla w_\varrho|^2\right| & \leq CL^2\wp_w \\
\left|\frac{\partial\mathcal{J}}{\partial\varphi}\triangle w_\varphi\right| & \leq C\wp^5\frac{CL^2}{\wp_w^4} \\
& \leq CL\wp^2 \\
\left|\frac{\partial^2\mathcal{J}}{\partial\varphi^2}|\nabla w_\varphi|^2\right| & \leq CL\wp_w^2 \\
\left|\frac{\partial^2\mathcal{J}}{\partial\varphi\partial\varrho}\nabla w_\varrho\cdot\nabla w_\varphi\right| & \leq CL^2.
\end{aligned}
$$

Thus, we conclude

$$
\triangle\mathcal{J}(w_\varrho, w_\varphi) \leq C_0 \text{ in } B_R(0). \tag{87}
$$

where C_0 denotes a generic constant dependent ony on E_0, R and $\wp_w$. Similarly,

$$
\begin{aligned}
\left|\triangle\frac{\partial\mathcal{J}}{\partial\varphi}(w_\varrho, w_\varphi)\right| &= \left|\frac{\partial^2\mathcal{J}}{\partial\varphi\partial\varrho}\triangle w_\varrho + \frac{\partial^3\mathcal{J}}{\partial\varphi\partial\varrho^2}|\nabla w_\varrho|^2 + \frac{\partial^2\mathcal{J}}{\partial\varphi^2}\triangle w_\varphi + \frac{\partial^3\mathcal{J}}{\partial\varphi^3}|\nabla w_\varphi|^2\right. \\
&\quad \left. +2\frac{\partial^3\mathcal{J}}{\partial\varphi^2\partial\varrho}\nabla w_\varrho\cdot\nabla w_\varphi\right| \\
&\leq C_0 \text{ in } B_R(0),
\end{aligned} \tag{88}
$$

$$
\begin{aligned}
\left|\nabla\frac{\partial\mathcal{J}}{\partial\varphi}(w_\varrho, w_\varphi)\cdot\nabla w_\varphi\right| &= \left|\frac{\partial^2\mathcal{J}}{\partial\varrho\partial\varphi}(w_\varrho, w_\varphi)\nabla w_\varrho\cdot\nabla w_\varphi + \frac{\partial^2\mathcal{J}}{\partial\varphi^2}(w_\varrho, w_\varphi)|\nabla w_\varphi|^2\right| \\
&\leq C\wp^4\frac{L^2}{\wp^3} + C\wp^7\frac{L^2}{\wp^6} \\
&\leq C_0 \text{ in } B_R(0)
\end{aligned} \tag{89}
$$

and

$$
\begin{aligned}
\left|\frac{\partial\mathcal{J}}{\partial\varphi}(w_\varrho, w_\varphi)\triangle w_\varphi\right| &\leq C\wp_w^5\frac{CL^2}{\wp_w^4} \\
&\leq C_0 \text{ in } B_R(0).
\end{aligned} \tag{90}
$$

By (87), (88), (89) and (90),

$$
|\triangle w_\Phi| \leq C_0\varphi_0 \text{ in } B_R(0). \tag{91}
$$

Thus, by (84), (91) and elliptic regularity, for any $\alpha \in (0,1)$

$$|w_\Phi|_{C^{1,\alpha}(B_{\frac{15R}{16}}(0))} \leq C_0 \left(|\triangle w_\Phi|_{L^\infty(B_R(0))} + |w_\Phi|_{L^\infty(B_R(0))} \right) \leq C_0 \phi_0. \quad (92)$$

Since

$$\nabla w_\Phi = \mathcal{J}(w_\varrho, w_\varphi) \nabla w_\varphi + \frac{\partial \mathcal{J}}{\partial \varrho}(w_\varrho, w_\varphi) \nabla w_\varrho w_\varphi + \frac{\partial \mathcal{J}}{\partial \varphi}(w_\varrho, w_\varphi) \nabla w_\varphi w_\varphi,$$

we have

$$|\mathcal{J}(w_\varrho, w_\varphi) \nabla w_\phi| \leq \left| \frac{\partial \mathcal{J}}{\partial \varrho}(w_\varrho, w_\varphi) \nabla w_\varrho w_\varphi \right| + \left| \frac{\partial \mathcal{J}}{\partial \varphi}(w_\varrho, w_\varphi) \nabla w_\varphi w_\varphi \right| + |\nabla w_\Phi|.$$

By Lemma 8, Lemma 9 and (80), the first two terms are bounded by $C_0 \phi_0$. By (92), the third term is also bounded by $C_0 \phi_0$. Thus, we obtain first estimate of the lemma. Combined with (75), we also obtain

$$\wp_w^3 |\nabla w_\varphi| \leq C_0 \varphi_0 \quad \text{in } B_{\frac{15R}{16}}(0). \quad (93)$$

Next, we compute

$$\begin{aligned}
\triangle w_\Upsilon = {} & \triangle w_\varrho + \Gamma^\varrho_{\varphi\varphi}(w_\varrho, w_\varphi) |\nabla w_\varphi|^2 \\
& + w_\varphi \Gamma^\varrho_{\varphi\varphi}(w_\varrho, w_\varphi) \triangle w_\varphi + \frac{w_\varphi^2}{2} \frac{\partial}{\partial \varphi} \Gamma^\varrho_{\varphi\varphi}(w_\varrho, w_\varphi) \triangle w_\varphi \\
& + \frac{w_\varphi^2}{2} \frac{\partial^2}{\partial \varphi^2} \Gamma^\varrho_{\varphi\varphi}(w_\varrho, w_\varphi) |\nabla w_\varphi|^2 + \frac{w_\varphi^2}{2} \frac{\partial}{\partial \varrho} \Gamma^\varrho_{\varphi\varphi}(w_\varrho, w_\varphi) \triangle w_\varrho \\
& + \frac{w_\varphi^2}{2} \frac{\partial^2}{\partial \varrho^2} \Gamma^\varrho_{\varphi\varphi}(w_\varrho, w_\varphi) |\nabla w_\varrho|^2 + w_\varphi \frac{\partial}{\partial \varrho} \Gamma^\varrho_{\varphi\varphi}(w_\varrho, w_\varphi) \nabla w_\varrho \cdot \nabla w_\varphi \\
& + \frac{w_\varphi^2}{2} \frac{\partial^2}{\partial \varrho \partial \varphi} \Gamma^\varrho_{\varphi\varphi}(w_\varrho, w_\varphi) \nabla w_\varrho \cdot \nabla w_\varphi. \quad (94)
\end{aligned}$$

The harmonic map equation (81) implies that the first two terms of (94) cancel. Combining with (78), (83), (93) and the Christoffel symbols estimates in the previous section, we obtain bounds for the other terms on the right hand side of (94). More precisely,

$$\begin{aligned}
\left| w_\varphi \Gamma^\varrho_{\varphi\varphi}(w_\varrho, w_\varphi) \triangle w_\varphi \right| &\leq C \varphi_0 \wp_w^5 \left((\wp_w^{-1} |\nabla w_\varrho| |\nabla w_\varphi| + \wp_w^2 |\nabla w_\varphi|^2 \right) \\
&\leq C \varphi_0^2 \wp_w L^2, \\
\left| \frac{w_\varphi^2}{2} \frac{\partial}{\partial \varphi} \Gamma^\varrho_{\varphi\varphi}(w_\varrho, w_\varphi) \triangle w_\varphi \right| &\leq C \varphi_0^2 \wp_w^7 (w_\varrho, w_\varphi) \left(\wp_w^{-1} |\nabla w_\varrho| |\nabla w_\varphi| + \wp_w^2 |\nabla w_\varphi|^2 \right)
\end{aligned}$$

$$
\begin{aligned}
& \leq C\varphi_0^2\wp_w^3 L^2, \\
\left|\frac{w_\varphi^2}{2}\frac{\partial^2}{\partial\varphi^2}\Gamma^\varrho_{\varphi\varphi}(w_\varrho, w_\varphi)|\nabla w_\varphi|^2\right| & \leq C\varphi_0^2\wp_w^6|\nabla w_\varphi|^2 \\
& \leq C\varphi_0^2 L^2, \\
\left|\frac{w_\varphi^2}{2}\frac{\partial}{\partial\varrho}\Gamma^\varrho_{\varphi\varphi}(w_\varrho, w_\varphi)\triangle w_\varrho\right| & \leq C\varphi_0^2\wp_w^4\cdot\wp_w^5|\nabla w_\varphi|^2 \\
& \leq C\varphi_0^2\wp_w^3 L^2, \\
\left|\frac{w_\varphi^2}{2}\frac{\partial^2}{\partial\varrho^2}\Gamma^\varrho_{\varphi\varphi}(w_\varrho, w_\varphi)|\nabla w_\varrho|^2\right| & \leq C\varphi_0^2\wp_w^3|\nabla w_\varrho|^2 \\
& \leq C\varphi_0^2\wp_w^3 L^2, \\
\left|w_\varphi\frac{\partial}{\partial\varrho}\Gamma^\varrho_{\varphi\varphi}(w_\varrho, w_\varphi)\nabla w_\varrho\cdot\nabla w_\varphi\right| & \leq C\varphi_0\wp_w^4|\nabla w_\varrho||\nabla w_\varphi| \\
& \leq C\varphi_0^2\wp_w L^2, \\
\left|\frac{w_\varphi^2}{2}\frac{\partial^2}{\partial\varrho\partial\varphi}\Gamma^\varrho_{\varphi\varphi}(w_\varrho, w_\varphi)\nabla w_\varrho\cdot\nabla w_\varphi\right| & \leq C\varphi_0^2\wp_w^5|\nabla w_\varrho||\nabla w_\varphi| \\
& \leq C\varphi_0^2\wp_w^2 L^2.
\end{aligned}
$$

In summary, we have shown that there exists a constant $C_0 > 0$ depending only on R the total energy of w such that

$$
|\triangle w_\Upsilon| \leq C_0\varphi_0^2 \text{ in } B_{\frac{15R}{16}}(0) \tag{95}
$$

which is the second estimate of the lemma. Q.E.D.

Definition 15 We say that a map $l = (l_\rho, l_\phi) : B_1(0) \to \overline{\mathbf{H}}_2$ is an *almost affine map* if $l_\rho(x) = \vec{a}\cdot x + b$ for $\vec{a} \in \mathbf{R}^n$ and $b \in \mathbf{R}$, i.e. l_ρ is an affine function.

Theorem 16 *Let $R \in [\frac{1}{2}, 1)$, $E_0 > 0$, $A_0 > 0$ and a normalized metric g on $B_R(0)$ (cf. Definition 13) be given. There exist $C > 0$ and $\alpha > 0$ with the following property:*

For $\phi_0 > 0$ and $s \in (0, 1]$, if $w = (w_\rho, w_\phi) : (B_R(0), g_s) \to \overline{\mathbf{H}}[\varphi_0]$ is a harmonic map with

$$
E^w \leq E_0 \quad \textit{and} \quad |w_\rho|_{L^\infty(B_{\frac{15R}{16}}(0))} \leq A_0,
$$

then

$$\sup_{B_r(0)} d(w,\hat{l}) \leq Cr^{1+\alpha} \sup_{B_R(0)} d(w,L) + Cr\phi_0^2, \ \forall r \in (0,\frac{R}{2}]$$

where $\hat{l} = (\hat{l}_\rho, \hat{l}_\phi) : B_1(0) \to \overline{\mathbf{H}}$ *is the almost affine map given by*

$$\hat{l}_\rho(x) = w_\rho(0) + \nabla w_\rho(0) \cdot x, \quad \hat{l}_\phi(x) = w_\phi(x)$$

and $L : B_1(0) \to \overline{\mathbf{H}}_2$ *is any almost affine map.*

Proof. Lemma 14 is analogous to [DM1] Lemma 8. With this, the theorem follows from the proof of [DM1] Theorem 10. Q.E.D.

With Lemma 14, we are in position to prove the main theorem (cf. Theorem 18) of this section. We first start with the following definition.

Definition 17 For a constant $E_0 > 0$ and a normalized metric g on $B_1(0)$ (cf. Definition 13), define $\mathcal{F}[E_0, g]$ to be a set of harmonic maps $u : (B_1(0), g_\sigma) \to \overline{\mathbf{H}}_2$ with $\sigma \in (0,1]$, $u(0) = P_0$ and $E^u \leq E_0$.

Theorem 18 *Let* $E_0 > 0$, $A > 0$, $\delta_0 > 0$ *and a normalized metric* g *on* $B_1(0)$ *be given. There exists* $\sigma_0 > 0$ *and* $D_0 > 0$ *such that if* $\sigma \in (0, \sigma_0)$ *and* $u : (B_1(0), g_\sigma) \to \overline{\mathbf{H}}_2 \in \mathcal{F}[E_0, g]$ *(cf. Definition 17) satisfies*

$$\sup_{B_1(0)} d(u,l) < D_0 \tag{96}$$

where $l(x) = (Ax^1, 0)$ *when written in terms of coordinates* (ϱ, φ) *of (70), then*

$$s^{-1} \sup_{B_s(0)} |u_\varrho - Ax^1| < \delta_0.$$

Furthermore, for $\mathcal{L} = \{(\varrho, \varphi) \in \overline{\mathbf{H}}_2 : \varphi = 0\}$, *we have*

$$\lim_{s \to 0} s^{-1} \sup_{B_s(0)} d(u, \mathcal{L}) = 0. \tag{97}$$

Proof. The statement and the proof of [DM1] Lemma 14 can be modified to the present setting by simply replacing ϵ_0^3 by $C\wp(\epsilon_0, \phi_0)$. Thus, a straightforward modification of the proof of [DM1] Lemma 16 and [DM1] Theorem 17 implies the assertion. Q.E.D.

6 Proof of the main theorem

In this section, we provide the proof of Theorem 1. We first prove

Theorem 19 *Let g be a normalized metric on $B_1(0)$ (cf. Definition 13) and $u : (B_1(0), g) \to \overline{\mathbf{H}}$ be a harmonic map. If $u(0) = P_0$, then $Ord^u(0) \neq 1$*

Proof. Assume on the contrary that $Ord^u(0) = 1$ and $u(0) = P_0$. By the normalization (7), the energy of the blow up map $u_\sigma : (B_1(0), g_\sigma) \to \overline{\mathbf{H}}$ of u at $x_0 = 0$ is bounded by 2 for sufficiently small $\sigma > 0$. Lemma 5 implies that by choosing an the appropriate t_* (cf. (69)) to define coordinates (ϱ, φ) (cf. (70)) and rotating if necessary, $u_\sigma \in \mathcal{F}[2, g]$ satisfies assumption (96) of Theorem 18 for some sufficiently small $\sigma > 0$. Thus, (97) implies that for $\mathcal{L} = \{(\varrho, \varphi) \in \overline{\mathbf{H}}_2 : \varphi = 0\}$, we have

$$\lim_{s\to 0} d(P_0, s^{-1}\mathcal{L}) = \lim_{s\to 0} s^{-1} d(P_0, \mathcal{L}) = \lim_{s\to 0} s^{-1} d(u_\sigma(0), \mathcal{L}) = 0.$$

This is a contradiction. Indeed, $\mathcal{L}$ does not contain the point P_0, and hence

$$\lim_{s\to 0} d(P_0, s^{-1}\mathcal{L}) = \infty.$$

Q.E.D.

Proof of Theorem 1. By [DM2] Theorem 16,

$$\dim_{\mathcal{H}}\{x \in \Omega : Ord^u(x) > 1\} \leq n - 2.$$

Thus, Theorem 19 implies the assertion. Q.E.D.

The motivation of this paper is in the study of harmonic maps into the metric completion $\overline{\mathcal{T}}_{g,n}$of the Teichmüller space with respect to the Weil-Petersson metric. With this in mind, we now extend Theorem 19 to the following slightly more general setting. Let $\mathbf{H}^{k-j}$ (resp. $\overline{\mathbf{H}}^{k-j}$) be the product space of $(k-j)$-copies of $\mathbf{H}$ (resp. $\overline{\mathbf{H}}$). Let

$$h = g_{\mathbf{H}} \oplus \ldots \oplus g_{\mathbf{H}} \quad \text{and} \quad d = d_h$$

denote the product metric on $\mathbf{H}^{k-j}$ and the distance function on $\overline{\mathbf{H}}^{k-j}$ induced by h respectively. Define

$$\mathcal{P}_0 := (P_0, \ldots, P_0) \in \overline{\mathbf{H}}^{k-j}.$$

The boundary $\partial\overline{\mathbf{H}}^{k-j}$ of $\overline{\mathbf{H}}^{k-j}$ the set of points $(P^1,\dots,P^{k-j})\in\overline{\mathbf{H}}^{k-j}$ such that at least one of $P^1,\dots,P^{k-j}$ is equal to P_0. Define

$$\lambda(P^1,\dots,P^{k-j}):=(\lambda P^1,\dots,\lambda P^{k-j}).$$

Given a map $v:B_1(0)\to\overline{\mathbf{H}}^{k-j}$, define its blow up maps

$$v_\sigma:B_1(0)\to\overline{\mathbf{H}}$$

by setting

$$I^v(\sigma)=\int_{\partial B_\sigma(0)}d^2(v,P_0)d\Sigma,\quad \nu(\sigma)=\sqrt{\frac{I^v(\sigma)}{\sigma^{n-1}}}.$$

and

$$v_\sigma:B_1(0)\to\overline{\mathbf{H}}^{k-j},\quad v_\sigma(x)=\nu^{-1}(\sigma)v(\sigma x).$$

A harmonic map $u:B_1(0)\to\overline{\mathcal{T}}_{g,n}$ can be written locally as $u=(V,v)$ where $V:B_1(0)\to\mathbf{R}^{2j}$ and $v:B_1(0)\to\overline{\mathbf{H}}^{k-j}$. The maps V, v are not harmonic maps, but they are only *approximately harmonic* since the Weil-Petersson metric and the Weil-Petersson connection are *asymptotically a product* near the boundary. This will be explained in greater detail in the upcoming paper [DMW]. For now, we will prove the following theorem concerning a map $v:B_1(0)\to\overline{\mathbf{H}}^{k-j}$ that is not necessarily harmonic but that shares important properties (cf. (i) and (ii) below) with it (with $C=0$ in inequality (98) below).

Theorem 20 *There does not exists a map $v:B_1(0)\to\overline{\mathbf{H}}^{k-j}$ with $v(0)=\mathcal{P}_0$ satisfying the following:*

- (i) *There exists a sequence $\sigma_i\to 0$ such that v_{σ_i} converges locally uniformly in the pullback sense to a linear function.*
- (ii) *There exists $\sigma_0>0$ such that for any $\sigma\in(0,\sigma_0]$ and a harmonic map $w:(B_1(0),g_\sigma)\to\overline{\mathbf{H}}$, there exists a constant $C>0$ such that*

$$\sup_{B_{\frac{15R}{16}}(0)}d^2(v_\sigma,w)\le c_0\int_{\partial B_R(0)}d^2(v_\sigma,w)d\Sigma_{g_\sigma}+C\sigma^2. \tag{98}$$

Proof. After a rotation of the domain if necessary, we may assume that the sequence v_{σ_i} converges locally uniformly in the pullback sense to a linear function $L(x)=Ax^1$. By applying [DM1] Lemma 7

to $B_1^{\pm}(0) = \{x \in B_1(0) : x^1 \geq (\leq)\ 0\}$ and following the argument of Lemma 5, we can reach the same conclusion of Lemma 5 with v_{σ_i} replacing u_{σ_i}. As discussed in [DM1] Remark 18, Assumption (ii) can take the place of harmonicity in [DM1] Lemma 16 and [DM1] Theorem 17. Theorem. Thus, we can also do this in Theorem corofinduct above. Now we can argue as in proof of Theorem 19 to obtain the result. Q.E.D.

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