Regularity theory of harmonic maps between singular spaces

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Abstract

We discuss regularity issues for harmonic maps from an $n$-dimensional Riemannian polyhedral complex $X$ to a non-positively curved metric space. The main theorems assert, assuming Lipschitz regularity of the metric on the domain complex, that such maps are locally Hölder continuous with explicit bounds of the Hölder constant and exponent on the energy of the map and the geometry of the domain and locally Lipschitz continuous away from the $(n-2)$-skeleton of the complex. Moreover, if $x$ is a point on the $k$-skeleton we give explicit dependence of the Hölder exponent at a point near $x$ on the combinatorial and geometric information of the link of $x$ in $X$ and the link of the $k$-dimensional skeleton in $X$ at $x$.

1 Introduction

The seminal work of M. Gromov and R. Schoen [GS] extends the study of harmonic maps between smooth manifolds to the case when the target is a Riemannian simplicial complex of non-positive curvature. The theory of harmonic maps into singular spaces was expanded substantially by the work of N. Korevaar and R. Schoen [KSI] and [KS2] where they consider targets that are arbitrary metric spaces of non-positive curvature. (Such spaces are called NPC or CAT(0) if they are simply connected.) One important motivation for considering singular spaces in the theory of harmonic maps is in studying group representations. The main application of the Gromov-Schoen theory is to establish a certain case of non-Archimedean superrigidity complementing Corlette’s Archimedean superrigidity for lattices in groups of real rank 1 [Co].

The next step in the generalization of the harmonic map theory is to replace smooth domains by singular ones. This problem is also motivated by superrigidity, in this case when the domain group is non-Archimedean. The consideration
of a Riemannian simplicial complex as the domain space for harmonic maps seems to have been initiated by J. Chen [Ch]. Subsequently, this theory was further elaborated by J. Eells and B. Fuglede [EF] and [F]. In particular, they show Hölder continuity for harmonic maps under an appropriate smoothness assumption for the metric on each simplex.

The development of the harmonic map theory from a Riemannian complex is important in the study of non-Archimedean lattices. Considering a 2-dimensional domains [DM1], [DM2] and [DM3] establish fixed point and rigidity theorems of harmonic maps from certain flat 2-complexes. The key issue in the techniques introduced in these papers is to prove regularity theorems strong enough to be able to apply differential geometric methods.

Recall that the main idea of [GS] is also to show that harmonic maps are regular enough so that Bochner methods could be used in the setting of singular targets. In particular, the fundamental regularity result of [GS] and of [KS1] is that harmonic maps from a smooth Riemannian domain into a NPC target are locally Lipschitz continuous. As noted in [Ch], this statement no longer holds when we replace the domain by a polyhedral space. On the other hand, we have found in [DM2] and [DM3] that modulus of continuity better than Hölder is crucial in applications. This necessitates stronger regularity results than Hölder.

This paper is meant to be the state of the art in the regularity theory of harmonic maps from Riemannian cell complexes to non-positively curved metric spaces (cf. Section 2 for precise definitions). Our first theorem concerns Hölder continuity of harmonic maps. This is a generalization of the result of [EF] for Lipschitz metrics.

**Theorem** (cf. Theorem 30) Let $B(r)$ be a ball or radius $r$ around a point in an admissible complex $X$ endowed with a uniformly elliptic Lipschitz metric $g$. If $(f : B(r), g) \to Y$ is a harmonic map, then there exist $C$ and $\gamma$ so that

$$d(f(x), f(y)) \leq C|x - y|^\gamma \quad \forall x, y \in B(r/2).$$

Here, $C$ and $\gamma$ only depend on the Lipschitz bound and the ellipticity constant of $g$, the energy of the map $E_f$ and the geometry of $B(r)$.

Note that our approach to Hölder continuity follows the one in [GS] and [Ch] and is completely different from the one in [EF] and [F]. In our case a variant of the Gromov-Schoen monotonicity formula allows us to obtain energy decay estimates which in turns imply Hölder continuity by an adaptation of an argument due to Morrey. The technical difficulty is that we make no assumption that the boundary of each simplex is totally geodesic as it is implicitly assumed in [Ch]. Our method also differs from the one in [GS] due to the fact that for singular domains the monotonicity formula does not hold for large balls (cf. Remark 3.3 of [DM1]). The main technical hurdle is to obtain energy decay estimate with uniform radius and this is handled in Section 4.
Our main theorem concerns better regularity of harmonic maps. More precisely, we show that harmonic maps are Lipschitz continuous away from the codimension 2 skeleton $X^{(n-2)}$ of $X$. For points that lie on the lower dimensional skeleta we also give an estimate of the Hölder exponent of the harmonic map in terms of the first eigenvalue of the link of the normal stratum of the skeleton. More precisely, let $x \in X^{(k)} - X^{(k-1)}$ and let $N = N(x)$ denote the link of $X^{(k)}$ at $x$ along with the metric induced by the given Lipschitz Riemannian metric. Note $N$ is a spherical $(n-k-1)$-complex. Set

$$
\lambda_1^N := \inf_{Q \in Y} \lambda_1(N, T_Q Y),
$$

where $\lambda_1(N, T_Q Y)$ denotes the first eigenvalue of the Laplacian of $N$ with values in the tangent cone of $Y$ at $Q$. (See Section 8 for further details.)

More precisely, our main theorem is as follows:

**Main Theorem** (cf. Theorem 63) Let $B(r)$ be a ball or radius $r$ around a point $x$ in an admissible complex $X$ endowed with a uniformly elliptic Lipschitz metric $g$, $(Y, d)$ a NPC space and $f : (B(r), g) \to Y$ a harmonic map.

1. If $x \in X - X^{(n-2)}$ let $d$ denote the distance of $x$ to $X^{(n-2)}$. Then for any $d' < \min(r, \frac{d}{2})$, $f$ is Lipschitz continuous in $B(d')$ with Lipschitz constant dependent on the Lipschitz bound and the ellipticity constants of $g$, $E_f$ and $d$.

2. If $x \in X^{(k)} - X^{(k-1)}$ for $k = 0, ..., n-2$ let $d$ denote the distance of $x$ to $X^{(k-1)}$. Then, for any $d' < \min(r, d/2)$, $f$ is Hölder continuous in $B(d')$ with Hölder exponent and constant dependent on the Lipschitz bound and the ellipticity constant of $g$, $E_f$ and $d$. More precisely, the Hölder exponent $\alpha$ has a lower bound given by the following: If $\lambda_1^N \geq \beta$ then $\alpha(\alpha + n - k - 2) \geq \beta$. In particular, if $\lambda_1^N \geq n-k-1$, then $f$ is Lipschitz continuous in a neighborhood of $x$.

The paper is organized as follows: In Section 2 we define our domain and target spaces and recall the notion of harmonic maps. In Section 3 we prove the monotonicity formula in our setting and in Section 4 we discuss the Hölder continuity of harmonic maps. Section 3 is in some sense the heart of the paper as all subsequent results depend on it. Though similar in spirit with the monotonicity formula of [GS] it also differs significantly in the fact that we show that the relevant quantity (the order function) is not monotone as in [GS]. Nonetheless, we show that the order function has a well defined limit. This is necessary due to the fact that the different strata in the complex are not assumed to be totally geodesic. As mentioned before in Section 4 we are including a proof of the Hölder continuity of harmonic maps with a slightly more relaxed assumption on the metric of the complex than in [EF] and [F].

The purpose of Sections 5 and 6 is to construct a tangent map. We then establish properties of maps from a flat domain in order to analytize the tangent map in section 7. Finally, Section 8 is devoted to the proof of the Main Theorem.
2 Domain and target spaces

2.1 Local models

We now introduce our local models which will represent a neighborhood of a point in a complex. A half space is a connected component $H$ of $\mathbb{R}^n - h$ where $h$ is a hyperplane. By a normalized half space, we will mean a half space $H$ so that the hyperplane $h$ that defines $H$ contains the origin $\vec{0}$. We say the normalized half spaces $H_1, ..., H_\nu$ are linearly independent if the normals to the hyperplanes $h_1, ..., h_\nu$ defining the half spaces are linearly independent. A wedge (or a $\nu$-wedge) $W$ is the closure of the intersection of $\nu$ number of linearly independent normalized half spaces $H_1, ..., H_\nu$. By its construction, every $\nu$-wedge $W$ is a $n$-dimensional cone in $\mathbb{R}^n$ with $\vec{0}$ as the vertex. Wedge angles are the angles between any pair of vectors $h_1, ..., h_\nu$. In particular a 2-wedge has one wedge angle, and in general a $\nu$-wedge has $\frac{\nu(\nu-1)}{2}$ number of wedge angles. A face of the wedge is an intersection $W \cap h_i \cap ... \cap h_j$, $1 \leq i_1 \leq ... \leq i_j \leq \nu$. For example, the intersection $W \cap h_1 \cap ... \cap h_\nu$ is a face which is a $(n-\nu)$-dimensional linear subspace of $\mathbb{R}^n$. This is the lowest dimensional face and we denote it by $D$. We will use the coordinates of $\mathbb{R}^n$ to label points in $W$. For simplicity, we always choose the coordinate system $(x^1, ..., x^n)$ of $\mathbb{R}^n$ so that $D$ is given as $x^{n-\nu+1} = ... = x^n = 0$.

A local model $B$ is a disjoint union of wedges $W_1, ..., W_l$ glued along their faces. More precisely, a gluing map is a linear isometry $\varphi : F \to F'$ where $F$ and $F'$ are faces of $W_i$ and $W_j$, $1 \leq i, j \leq l$. We say $x \sim x'$ if $\varphi(x) = x'$ and we let $B = \cup W_i/\sim$. Given a face $F$ of a wedge $W$, we will also call its equivalence class in $B$ a face. A face of $B$ is said to belong to a wedge $W$ if its equivalence class contains a face of $W$. The boundary of a local model is the union of all $(n-1)$-dimensional faces which belong to exactly one wedge. A local model constructed from $\nu$-wedges will be referred to as a dimension-$n$, codimension-$\nu$ local model. The one exception to this terminology will be when we have two 1-wedges, i.e. two half spaces, glued together along $D$. In this case, the local model is simply $\mathbb{R}^n$ and this will be referred to as a codimension-0 local model.

Throughout the paper, we assume that our local models $B$ are admissible and without boundary. Admissible means that $B - F$ is connected for any $(n-2)$-dimensional face $F$. Since $W$ is a subset of $\mathbb{R}^n$, there is a natural Euclidean metric inherited from $\mathbb{R}^n$. This defines an Euclidean metric $\delta$ on $B$. For $x, y \in B$, let $|x - y|$ be the induced distance function from $\delta$. Set $B(r)$ be the $r$-ball centered at the origin of $B$ and $W(r) = B \cap W$ for any wedge $W$ of $B$. For the sake of simplicity, we will also refer to $W(r)$ as a wedge (of $B(r)$).

We now give examples of wedges in dimension 2 and dimension 3. (i) The only two dimensional 1-wedge is the half plane $\{ (x,y) \in \mathbb{R}^2 : y \geq 0 \}$. We consider a model space $B$ where $k$ copies of 1-wedges are glued together along $D = \{ (x,y) \in \mathbb{R}^2 : y = 0 \}$. This example models edge points of a two dimensional simplicial complex. (ii) An example of a two dimensional 2-wedge
is the first quadrant \( \{(x,y) \in \mathbb{R}^2 : x,y \geq 0 \} \). Another example is the set \( W = \{(x,y) \in \mathbb{R}^2 : \sqrt{3}x \geq y \geq 0 \} \). Note that \( D \) in this case is the point \( x = y = 0 \). A vertex point of a two dimensional simplicial complex can be modelled by a model space where \( l \) copies of \( W \) are glued together along their faces (in this case lines \( y = 0 \) or \( y = \sqrt{3}x \)) according to the combinatorial information of the complex. (iii) The only three dimensional 1-wedge is the half space \( \{(x,y,z) \in \mathbb{R}^3 : z \geq 0 \} \). The model space \( B \) where \( l \) copies of 1-wedges are glued together along \( D \) is of dimension \( 3 \). (iv) An example of a three dimensional 2-wedge is \( \{(x,y,z) \in \mathbb{R}^3 : y,z \geq 0 \} \). Another example is the set \( W = \{(x,y,z) \in \mathbb{R}^3 : \sqrt{3}y \geq z \geq 0 \} \). Here, \( D = \{(x,y,z) \in \mathbb{R}^3 : y = z = 0 \} \). A neighborhood of a point on a 1-skeleton of a three dimensional simplicial complex can be modelled by a model space \( B \) where \( l \) copies of \( W \) are glued together along their faces according to the combinatorial information. (v) An example of a three dimensional 3-wedge is the first octant \( \{(x,y,z) \in \mathbb{R}^3 : x,y \geq 0 \} \). Another example is the set \( W \) consisting of points of the form \( \sum_{i=1}^{3} t_i v_i, t_i \geq 0 \) where \( v_1 = (1,0,0), v_2 = (\frac{1}{2}, \frac{\sqrt{2}}{2}, 0) \) and \( v_3 = (\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}) \). Here, \( D \) is the point \((0,0,0)\). Note that the standard tetrahedron consists of points of the form \( \sum_{i=1}^{3} t_i v_i, 0 \leq t_i \leq 1 \). A neighborhood of a point on the 0-skeleton of a three dimensional simplicial complex can be modelled by a model space \( B \) where \( l \) copies of \( W \) are glued together along their faces according to the combinatorial information of the complex.

Let \( B \) be a dimension-\( n \), codimension-\( \nu \) local model and let \( \nu = n - k \). Recall that this means that \( D \subset B \) is of dimension \( k \); more specifically, \( D \) can be isometrically identified with \( \mathbb{R}^k \). We say \( x \in B \) is a dimension-\((n-j)\) singular point if \( B_\delta(x) \) is homeomorphic to \( B'(\delta) \) where \( B' \) is some dimension-\( n \), codimension-\((n-j)\) local model for some \( \delta > 0 \). We denote the closure of the set of codimension-\((n-j)\) singular points by \( S_j \) and set \( S_{-1} = \emptyset \). For example, if \( B \) is a dimension-\((n-k)\) local model, then \( S_k = D \) and \( S_i = \emptyset \) for \( i = -1, \ldots, k-1 \).

The following two definitions will be important in Section 4.

**Definition 1** Suppose \( x \in S_{j+1} - S_j \). Thus, \( x \) is an interior point of a \((j+1)\)-dimensional face \( F \). We define \( \pi_j(x) \) to be the set of all points \( x' \) in \( S_j \cap F \) so that \( d(x,x') = \min_{y \in S_j \cap F} d(x,y) \). First, note that the closest point projection of \( x \) to the boundary of \( F \) is not necessarily unique so that \( \pi_j(x) \) may contain more than one point. Secondly, because a face of a local model is a convex subset of Euclidean space, \( \pi_j(x) \subset S_j - S_{j-1} \). For \( i > j \), \( x \in S_i \) and \( x' \in S_j \), we write \( x \triangleright x' \) if we can arrive from \( x \) to \( x' \) by a sequence of successive projections, i.e. there exists a sequence

\[ x = y_i, y_{i-1}, \ldots, y_{j+1}, y_j = x' \]
so that

\[ y_i \in S_i, \]
\[ y_{i-1} \in \pi_{i-1}(y_i) \subset S_{i-1}, \]
\[ \ldots \]
\[ y_{j+1} \in \pi_{j+1}(y_{i-j-1}) \subset S_{j+1}, \]
\[ y_j \in \pi_j(y_{i-j}) \subset S_j. \]

For \( x \in B(\sigma) \), let \( \Pi_j(x) \) be the set of points \( x' \in S_j \) so that \( x \triangleright x' \). For any set \( N \), let \( \Pi_j(N) \) be the set of points \( x' \in S_j \) so that \( x \triangleright x' \) for \( x \in N \).

**Definition 2** For a point \( x \) in \( B \) with \( x \in S_{j+1} - S_j \) set

\[ R(x) = \min\{|x - y| : y \in S_j\}. \]

Note that this implies that \( R(x) \) is the radius of the largest homogeneous ball centered at \( x \) contained in \( B \).

In addition to the Euclidean metric \( \delta \) we equip a local model \( B \) (or \( B(r) \)) with a uniformly elliptic Lipschitz Riemannian metric \( g \). By this we mean that for each wedge \( W \) of \( B \) (resp. each face \( F \) of \( B \)), we have a Lipschitz Riemannian metric \( g_W \) (resp. \( g_F \)) up to the boundary of \( W \) (resp. \( F \)) with the property that \( g_W \) (resp. \( g_F \)) is uniformly elliptic with respect to \( \delta \) and if \( F' \) is a face of \( W \) (resp. \( F \)) then the restriction \( g_W \) (resp. \( g_F \)) to \( F' \) is equal to \( g_{F'} \). We say \( \lambda \in (0, 1] \) is an ellipticity constant if the ellipticity constants of \( g_W \) (resp. \( g_F \)) is bounded below by \( \lambda \). Note that we do not necessarily assume that the faces of the wedges are totally geodesic. We can express \( g \) as a matrix \((g_{\alpha\beta})\) in terms of the Euclidean coordinate system on the wedges \( W \) inherited from the Euclidean space.

**Definition 3** We say a metric \( g \) on \( B(r) \) is normalized if \( g_{\alpha\beta}(0) = \delta_{\alpha\beta} \).

### 2.2 Admissible Cell Complexes

A convex cell complex or simply a complex \( X \) in an affine space \( \mathbb{E}^d \) is a finite collection \( \{F^k\} \) of cells where each \( F^0 \) is a point, and each \( F^k \) is a bounded convex piecewise linear polyhedron with interior in some \( \mathbb{E}^k \subset \mathbb{E}^d \), such that the boundary \( \partial F^k \) of \( F^k \) is a union of \( F^s \) with \( s < k \) (called the faces of \( F^k \)), and such that if \( s < k \) and \( F^k \cap F^s \neq \emptyset \), then \( F^s \subset F^k \). For example a simplicial complex is a cell complex whose cells are all simplices. We will denote by \( X^{(i)} \) the \( i \)-dimensional skeleton of \( X \), i.e. the union of all cells \( F^k \) where \( k \leq i \). \( X \) is called \( n \)-dimensional or simply a \( n \)-complex if \( X^{(n+1)} = \emptyset \) but \( X^{(n)} \neq \emptyset \). The boundary of \( X \), denoted \( \partial X \), is the union of \( k \)-cells \( F^k \), \( k < n \), so that \( F \)
is a face exactly of exactly one $n$-cell. A point $p \in \partial X$ is called a boundary point and a point $p \in X - \partial X$ is called an interior point. In the sequel we will require the following conditions (sometimes called admissibility conditions, or admissible complex (cf. [EF]):

(1) $X$ is dimensionally homogeneous; i.e. for $k < n$, each $k$-cell is a face of a $n$-cell.

(2) $X$ is locally $(n-1)$-chainable; i.e. for every connected, open set $U \subset X$, the open set $U - X^{(n-2)}$ is connected.

A Lipschitz Riemannian $n$-complex is a convex cell complex where each cell $F$ is equipped with a uniformly elliptic Lipschitz Riemannian metric $g_F$ up to the boundary. We are assuming that if $F'$ is a face of $F$ then the restriction $g_F$ to $F'$ is equal to $g_{F'}$.

Admissible cell complexes are based on local models because of the following obvious proposition:

**Proposition 4** Let $X$ be an admissible Lipschitz Riemannian complex of dimension $n$ with metric $g$ given as $(g_{\alpha \beta})$. Let $x \in X^{(k)} - X^{(k-1)}$ and let $\lambda \in (0,1]$ be the ellipticity constant of $g$ near $x$. Then there exists a dimension-$n$, codimension-$(n-k)$ local model $B$ and a homeomorphism $L_x : B(\lambda R(x)) \to L_x(B(\lambda R(x))) \subset X$ so that

(i) $L_x(0) = x$

(ii) For any wedge $W$ of $B$, $L_x$ restricted to $W \cap B(\lambda R(x))$ maps into the closure $\overline{\sigma}$ of a $n$-simplex of $X$

(iii) with $W$ viewed as a subset of $\mathbb{R}^n$ as in Section 2.1, $L_x\big|_{W \cap B(\lambda R(x))}$ uniquely extends as an affine map $L_x$ defined on $\mathbb{R}^n$

(iv) the pullback metric $h = L_x^* g$ has the property that $h_{\alpha \beta}(0) = \delta_{\alpha \beta}$ with respect to the coordinate chart on $W$.

Because $g$ has ellipticity constant $\lambda$, $L_x$ maps the ball of radius $\lambda R(x)$ centered at $0$ into the largest ellipse contained in the ball of radius $R(x)$ centered at $x$.

We now mention briefly some examples of admissible cell complexes that the reader should have in mind. For more details we refer to [EF]:

(a) A triangulable Lipschitz manifold
(b) A normal complex analytic space
(c) A Bruhat-Tits building
For any finitely generated group $\Gamma$ there is a two dimensional admissible complex without boundary whose fundamental group is $\Gamma$ (cf. [DM1]).

### 2.3 Harmonic maps

We now define our target spaces.

**Definition 5** A complete metric space $(Y, d)$ is said to be a NPC (non-positively curved) space if the following conditions are satisfied:

(i) The space $(Y, d)$ is a length space. That is, for any two points $P$ and $Q$ in $Y$, there exists a rectifiable curve $\gamma_{PQ}$ so that the length of $\gamma_{PQ}$ is equal to $d(P,Q)$ (which we will sometimes denote by $d_{PQ}$ for simplicity). We call such distance realizing curves geodesics.

(ii) Let $P, Q, R \in Y$. Define $Q_t$ to be the point on the geodesic $\gamma_{QR}$ satisfying $d_{QQ_t} = td_{QR}$ and $d_{Q_tR} = (1-t)d_{QR}$. Then

$$d_{PQ_t}^2 \leq (1-t)d_{PQ}^2 + td_{PR}^2 - t(1-t)d_{QR}^2.$$

**Remark.** Simply connected Riemannian manifolds of non-positive sectional curvature, Bruhat-Tits Euclidean buildings associated with actions of $p$-adic Lie groups and $\mathbb{R}$-trees are examples of NPC spaces. These spaces are also referred to as CAT(0) spaces in literature. We refer to [BH] for more details.

We will now review the definition of harmonic maps. For details we refer the reader to [EF]. First, we define the energy of a map. Let $Y$ be a NPC space and $f : (B(r), g) \rightarrow Y$ be a $L^2$ map from the local model to $Y$. The energy $\mathcal{E} f$ is defined as the weak limit of the $\epsilon$-approximate energy density measures which are measures derived from the appropriate average difference quotients. More specifically, define the $\epsilon$-approximate energy $e_\epsilon : B(r) \rightarrow \mathbb{R}$ by

$$e_\epsilon(x) = \begin{cases} \int_{y \in S(x, \epsilon)} \frac{d^2(f(x), f(y))}{\epsilon^2} d\sigma_{x, \epsilon} & \text{for } x \in B(r) \setminus B(r)_\epsilon \\ 0 & \text{for } x \in B(r)_\epsilon \end{cases}$$

where $\sigma_{x, \epsilon}$ is the induced measure on the $\epsilon$-sphere $S(x, \epsilon)$ centered at $x$ and $B(r)_\epsilon = \{ x \in B(r) : d(x, \partial B(r)) > \epsilon \}$. Define a family of functionals $\mathcal{E}^f_\epsilon : C_c(B(r)) \rightarrow \mathbb{R}$ by setting

$$\mathcal{E}^f_\epsilon(\varphi) = \int_{B(r)} \varphi e_\epsilon d\mu_g.$$

**Definition 6** We say that $f : B(r) \rightarrow Y$ has finite energy (or that $f \in W^{1,2}(B(r), Y))$ if

$$\sup_{\varphi \in C_c(B(r)), 0 \leq \varphi \leq 1} \limsup_{\epsilon \to 0} \mathcal{E}^f_\epsilon(\varphi) < \infty.$$
Theorem 7 Suppose \( f : \mathcal{B}(r) \rightarrow \mathcal{Y} \) has finite energy. Then the measures \( e_\epsilon(x)dx \) converge weakly to a measure which is absolutely continuous with respect to the Lebesgue measure. Therefore, there exists a function \( e(x) \), which we call the energy density, so that \( e_\epsilon(x)d\mu_g \rightharpoonup e(x)d\mu_g \).

In analogy to the case of real valued functions, we write \( |\nabla f|^2_g(x) \) in place of \( e(x) \) (We will omit the subscript in \( |\nabla f|^2_g, d\mu_g \) etc. if it is clear which metric we are using). In particular, \( gE^f = \int_{\mathcal{B}(r)} |\nabla f|^2_g d\mu_g \).

Definition 8 A map \( f : \mathcal{B}(r) \rightarrow \mathcal{Y} \) is said to be harmonic if it is energy minimizing amongst all \( W^{1,2} \)-maps with the same boundary condition.

For a Lipschitz vector field \( V \) on \( \mathcal{B}(r) \), \( |f^*(-V)|_g^2 \) is similarly defined. The real valued \( L^1 \) function \( |f^*(-V)|_g^2 \) generalizes the norm squared on the directional derivative of \( f \).

Theorem 9 The operator \( g_{\pi^f} \) defined by

\[
g_{\pi^f}(V, W) = \frac{1}{2}|f^*(V + W)|_g^2 - \frac{1}{2}|f^*(V - W)|_g^2.
\]

is continuous, symmetric, bilinear, non-negative and tensorial. We call \( g_{\pi^f} \) the pull-back metric.

We refer to [KS1] and [EF] for more details. Let \( \{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \} \) be the standard Euclidean basis defined on each wedge inherited from \( \mathbb{R}^n \) and \( \delta \) be the standard Euclidean metric on each wedge. Set

\[
\frac{\partial f}{\partial x^i} \cdot \frac{\partial f}{\partial x^j} = g_{\pi^f} \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \quad \text{and} \quad \left| \frac{\partial f}{\partial x^i} \right|^2 = \frac{\partial f}{\partial x^i} \cdot \frac{\partial f}{\partial x^i}.
\]

Note that the energy \( gE^f \) is given by

\[
\int_{\mathcal{B}(r)} |\nabla f|_g^2 d\mu_g = \sum \int_{\mathcal{W}(r)} |\nabla f|_g^2 d\mu_g
\]

where the sum is taken over all wedges of \( \mathcal{B}(r) \) and

\[
|\nabla f|_g^2 = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x^i} \cdot \frac{\partial f}{\partial x^j}.
\]

For a set \( S \subset \mathcal{B} \), let

\[
\int_S |\nabla f|_g^2 d\mu_g.
\]

We end this section by stating the following Poincaré inequality due to Eells-Fuglede [EF] and [F].
Theorem 10 Let $X_0$ be a compact admissible complex with a uniformly elliptic Lipschitz Riemannian metric $g$. Let $(Y,d)$ be an NPC space. There is a constant $C$ depending only on $X_0$ and the metric $g$ so that for any $\varphi \in W^{1,2}(X_0,Y)$,

$$\inf_{P \in Y} \int_{X_0} d^2(\varphi, P)d\mu \leq C \, g^{E}\varphi.$$ 

3 Monotonicity formula

In this section, we prove a monotonicity formula for harmonic maps. This is a modified version of the monotonicity formula shown in [GS] where the domain space is a Riemannian manifold. The technical difficulties posed by the singular nature of the domain space considered in this paper is that we cannot necessarily work in normal coordinates and that the faces are not necessarily totally geodesic in wedges with respect to the metric given.

Let $B$ be a local model. We continue to use the Euclidean coordinates $(x^1, \ldots, x^n)$ in each wedge $W$. For $x, y \in B$, we denote the induced (Euclidean) distance by $|x - y|$. By definition, if $x = (x^1, \ldots, x^n)$ and $y = (y^1, \ldots, y^n)$ are on the same wedge of $B$, then $|x - y| = \sqrt{(x^1 - y^1)^2 + \ldots + (x^n - y^n)^2}$. Furthermore, we let $(r, \theta_1, \ldots, \theta_{n-1})$ be the corresponding polar coordinates, i.e. $r$ gives the radial distance from the origin and $\theta = (\theta_1, \ldots, \theta_{n-1})$ are the coordinates on the standard $(n-1)$-sphere. Let $g$ be a normalized Lipschitz metric defined on $B(r) = \{x \in B : |x| < r\}$, i.e. if $g = (g_{\alpha \beta})$ with respect to the coordinates $x = (x^1, \ldots, x^n)$ on a wedge $W$, then

$$|g_{\alpha \beta}(x) - g_{\alpha \beta}(\bar{x})| \leq c|x - \bar{x}|, \quad \forall x, \bar{x} \in W$$

and

$$|g_{\alpha \beta}(x) - \delta_{\alpha \beta}| \leq c\sigma \quad (1)$$

for $|x| \leq \sigma$. For $\sigma \in (0,r)$, we set

$$g^E f(\sigma) = \int_{B(\sigma)} |\nabla f|_g^2 d\mu_g \quad (2)$$

and

$$g^I f(\sigma) = \int_{\partial B(\sigma)} d^2(f, Q)d\Sigma_g \quad (3)$$

for some $Q \in Y$. Here $d\Sigma_g$ is the measure on $\partial B(\sigma)$ induced by $g$. For the rest of the section, we will use the notation

$$E(\sigma) = g^E f(\sigma) \quad \text{and} \quad I(\sigma) = g^I f(\sigma)$$

for simplicity.
If we assume that the domain is a Riemannian manifold and replace $B(\sigma)$ by a geodesic $\sigma$-ball, [GS] shows that

$$\sigma \mapsto e^{C\sigma} \frac{E(\sigma)}{I(\sigma)}$$

is a non-decreasing function where $C$ is some constant depending on the metric. Note that in our case $B(\sigma)$ is a $\sigma$-ball with respect to the Euclidean metric $\delta$ on $B$. The reason Euclidean balls are considered here is the possible incompatibility of the induced distance functions of the metrics given on two different wedges along a shared face. More specifically, let $g_1$ and $g_2$ be the metrics defined on wedges $W_1$ and $W_2$ sharing a face $F$. Since we do not assume $F$ is totally geodesic in $W_1$ or $W_2$, if $d_1$ and $d_2$ are the induced distance functions in $W_1$ and $W_2$ defined by $g_1$ and $g_2$ respectively, then $d_1$ does not necessarily equal $d_2$ in $F$.

We are thus considering a general Lipschitz metric $g$ with no restriction on the faces and this leads to a modified version of the monotonicity formula (cf. Theorem 17) which in turn gives a well defined version of the order (cf. Corollary 21.). For a model space $B$ with a Euclidean metric $\delta$, the monotonicity of (4) follows from [M1].

We say a continuous function $\eta$ defined on $B(r)$ is smooth if the restriction of $\eta$ to each wedge $W$ of $B(r)$ is smooth up to the boundary of $W$. The set of smooth functions with compact support in $B(r)$ will be denoted by $C^\infty_c(B(r))$. 

**Lemma 11** Let $f : (B(r), g) \to Y$ be a harmonic map. For any $\sigma \in (0, r)$ and $\eta \in C^\infty_c(B(\sigma))$,

$$\int_{B(\sigma)} \left( |\nabla f|^2_g (2 - n)\eta - |\nabla f|^2_g \sum \partial x^i \frac{\partial \eta}{\partial x^i} + 2 \sum g^{ik} \frac{\partial \eta}{\partial x^i} \frac{\partial f}{\partial x^j} \cdot \frac{\partial f}{\partial x^k} \right) d\mu_g + O(\sigma) E(\sigma) = 0$$

where $|O(\sigma)| \leq c\sigma$ and $c$ depends on the Lipschitz bound of $g$.

**Proof.** For $t$ sufficiently small, we define $F_t : B(r) \to B(r)$ by setting $F_t(x) = (1 + t\eta(x))x$ for each $x = (x^1, \ldots, x^n)$ in a wedge $W$. For $f_t : B(r) \to Y$ defined as $f_t = f \circ F_t$, a direct computation (cf. [GS] Section 2) on each wedge $W$ of $B(r)$ gives

$$\frac{d}{dt} E_f[W]|_{t=0}$$

$$= \int_{W} \left( |\nabla f|^2_g (2 - n)\eta - |\nabla f|^2_g \sum \partial x^i \frac{\partial \eta}{\partial x^i} + 2 \sum g^{ik} \frac{\partial \eta}{\partial x^i} \frac{\partial f}{\partial x^j} \cdot \frac{\partial f}{\partial x^k} \right) d\mu_g + \text{remainder}$$

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Here, the remainder term is given by

\[
\int_W \left( -\eta \sum_{i,j,k} \frac{\partial g^{ij}}{\partial x^k} x^k \frac{\partial f}{\partial x^i} \cdot \frac{\partial f}{\partial x^j} \sqrt{g} + |\nabla f|^2 \eta \sum_i x^i \frac{\partial \sqrt{g}}{\partial x^i} \right) \, dx.
\]

Since we assume the metric \( g \) is Lipschitz, there exists a constant \( c \) so that 
\[
\left| \frac{\partial g^{ij}}{\partial x^k} \right|, \left| \frac{\partial \sqrt{g}}{\partial x^i} \right| \leq c,
\]
which then implies that the remainder term is bounded by 
\[
c\sigma \times E(\sigma).
\]
Summing over all the wedges \( W \) of \( B(\sigma) \) we get the right hand side of (5) and this is equals 0 since \( f_0 = f \) is harmonic. q.e.d.

**Lemma 12** If \( f : (B(r), g) \to Y \) satisfies (5), then for \( \sigma \in (0, r) \)

\[
\left| \frac{E'(\sigma)}{E(\sigma)} - \frac{n-2}{\sigma} - \frac{2}{E(\sigma)} \int_{\partial B(\sigma)} g^{ik} x^i \frac{\partial f}{\partial x^k} \frac{\partial f}{\partial r} \, d\Sigma_g \right| \leq c_1
\]

(6)

for some constant \( c_1 \) depending on the Lipschitz bound of \( g \),

\[
E'(\sigma) = (1 + O(\sigma)) \left( \frac{n-2 + O(\sigma)}{\sigma} E(\sigma) + 2 \int_{\partial B(\sigma)} \frac{\left| \frac{\partial f}{\partial r} \right|^2}{\partial r} \, d\Sigma_g \right)
\]

(7)

and

\[
(1 + c\sigma) E'(\sigma) \geq \frac{n-2}{\sigma} + \frac{2}{E(\sigma)} \int_{\partial B(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 \, d\Sigma_g - c_1
\]

(8)

where \( c \) is as in (1). In the above inequalities, recall that

\[
\frac{\partial f}{\partial x^i} \cdot \frac{\partial f}{\partial r} = \delta r \left( \frac{\partial f}{\partial x^i}, \frac{\partial f}{\partial r} \right) \quad \text{and} \quad \left| \frac{\partial f}{\partial r} \right|^2 = \frac{\partial f}{\partial r} \cdot \frac{\partial f}{\partial r} = \delta r \left( \frac{\partial f}{\partial r}, \frac{\partial f}{\partial r} \right).
\]

**Proof.** Let \( \eta \) approximate the characteristic function of \( B(\sigma) \) to obtain

\[
E'(\sigma) - \frac{n-2 + O(\sigma)}{\sigma} E(\sigma) - 2 \int_{\partial B(\sigma)} g^{ik} x^i \frac{\partial f}{\partial x^k} \cdot \frac{\partial f}{\partial r} \, d\Sigma_g = 0
\]

(9)

which immediately implies inequality (6). Next, we use the inequality \( g^{ik} \leq \delta^{ik} + c\sigma \) to show

\[
\sum_{i,k} g^{ik} x^i \frac{\partial f}{\partial x^k} \cdot \frac{\partial f}{\partial r} \leq \left| \frac{\partial f}{\partial r} \right|^2 + c\sigma \sum_k \left| \frac{\partial f}{\partial x^k} \cdot \frac{\partial f}{\partial r} \right| \cdot \left| \frac{\partial f}{\partial r} \right| \leq \left| \frac{\partial f}{\partial r} \right|^2 + c\sigma |\nabla f|^2
\]

Using this, inequalities (7) and (8) follows again from (9). q.e.d.
Lemma 13 Let \( f : (B(r), g) \to Y \) be a harmonic map. For any \( Q \in Y \),
\[
\Delta d^2(f, Q) - 2|\nabla f|^2 \geq 0 \text{ weakly, i.e.}
\]
\[
2 \int_{B(r)} |\nabla f|^2 \eta \, d\mu_g \leq -\int_{B(r)} <\nabla d^2(f, Q), \nabla \eta>_g \, d\mu_g 
\]
(10)
for any \( \eta \in C_\infty c(B(r)) \).

Proof. This inequality follows from a target variation of the harmonic map and hence the singular nature of the domain is not essential in the proof. Details can be found in the proof of [GS] Proposition 2.2. Q.E.D.

Lemma 14 If \( f : (B(r), g) \to Y \) satisfies (10), then
\[
E(\sigma) \leq I(\sigma)^{\frac{1}{2}} \left( \left( \int_{\partial B(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 \, d\Sigma_g \right)^{\frac{1}{2}} + c\sigma(E'(\sigma))^{\frac{1}{2}} \right) 
\]
(11)
and
\[
2E(\sigma) \leq \int_{\partial B(\sigma)} \frac{\partial}{\partial r} d^2(f, Q) \, d\Sigma_g + I(\sigma) + k\sigma^2 E'(\sigma) 
\]
(12)
for some constants \( c, k \) depending on the Lipschitz bound of \( g \) and
\[
\frac{1}{I(\sigma)} \int_{\partial B(\sigma)} \frac{\partial}{\partial r} d^2(f, Q) \, d\Sigma_g \leq \frac{2}{E(\sigma)} \int_{\partial B(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 \, d\mu_g + 2c\sigma E'(\sigma) \frac{E(\sigma)}{E'(\sigma)} 
\]
(13)
where \( c \) is as in (1).

Proof. Let \( \eta \) in (10) approximate the characteristic function of \( B(\sigma) \) to obtain
\[
2E(\sigma) \leq \int_{\partial B(\sigma)} <\nabla d^2(f, Q), \nabla |x|>_g \, d\Sigma_g = \int_{\partial B(\sigma)} g^{ij} \frac{\partial}{\partial x^i} d^2(f, Q) \frac{x^j}{|x|} \, d\Sigma_g .
\]
Using the estimate \( g^{\alpha\beta} \leq \delta^{\alpha\beta} + c\sigma \), we obtain
\[
2E(\sigma) \leq \int_{\partial B(\sigma)} \frac{\partial}{\partial r} d^2(f, Q) \, d\Sigma_g + c\sigma \int_{B(\sigma)} \sum_i \left| \frac{\partial}{\partial x^i} d^2(f, Q) \right| \, d\Sigma_g 
\]
(14)
\[
\leq 2 \int_{\partial B(\sigma)} d(f, Q) \frac{\partial}{\partial r} d(f, Q) \, d\Sigma_g + 2c\sigma \int_{\partial B(\sigma)} d(f, Q) \sum_i \left| \frac{\partial}{\partial x^i} d(f, Q) \right| \, d\Sigma_g 
\]
\[
\leq 2I(\sigma)^{\frac{1}{2}} \left( \int_{\partial B(\sigma)} \left| \frac{\partial}{\partial r} d(f, Q) \right|^2 \, d\Sigma_g \right)^{\frac{1}{2}} 
\]
\[
+ 2c\sigma I(\sigma)^{\frac{1}{2}} \left( \int_{\partial B(\sigma)} \sum_i \left| \frac{\partial}{\partial x^i} d(f, Q) \right|^2 \, d\Sigma_g \right)^{\frac{1}{2}} .
\]
The triangle inequality implies that
\[
\left| \frac{\partial}{\partial x^i} d(f, Q) \right|^2 \leq \frac{\partial f}{\partial x^i} \cdot \frac{\partial f}{\partial x^i} =: \left| \frac{\partial f}{\partial r} d(f, Q) \right|^2 \quad \text{and} \quad \left| \frac{\partial f}{\partial x^i} \right|^2 \leq \left| \frac{\partial f}{\partial r} \right|^2.
\]

From this, (11) follows immediately. Additionally, use the Cauchy-Schwarz inequality to obtain
\[
2c_\sigma d(f, Q) \left| \frac{\partial}{\partial x^i} d(f, Q) \right| \leq d^2(f, Q) + c^2_\sigma \left| \frac{\partial f}{\partial x^i} \right|
\]
which implies
\[
2E(\sigma) \leq \int_{\partial B(\sigma)} \frac{\partial}{\partial r} d^2(f, Q) d\Sigma_g + \int_{\partial B(\sigma)} d^2(f, Q) d\Sigma_g + c^2_\sigma \int_{\partial B(\sigma)} |\nabla f|^2 d\Sigma_g.
\]
From this, (12) follows immediately. Lastly, again use (14) to obtain
\[
E(\sigma) \int_{\partial B(\sigma)} \frac{\partial}{\partial r} d^2(f, Q) d\Sigma_g
\]
\[
\leq 2 \left( \int_{\partial B(\sigma)} d(f, Q) \frac{\partial}{\partial r} d(f, Q) d\Sigma_g \right)^2
\]
\[
+ 2c_\sigma \left( \int_{\partial B(\sigma)} d(f, Q) \frac{\partial}{\partial r} d(f, Q) d\Sigma_g \right) \left( \int_{\partial B(\sigma)} d(f, Q) \sum_i \left| \frac{\partial}{\partial x^i} d(f, Q) \right| d\Sigma_g \right)
\]
\[
\leq 2I(\sigma) \int_{\partial B(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\Sigma_g + 2c_\sigma I(\sigma) \int_{\partial B(\sigma)} |\nabla f|^2 d\Sigma_g.
\]
This immediately implies (13). Q.E.D.

The following energy growth estimate is also given in [Ch] with geodesic balls (and not Euclidean balls as it is here).

**Lemma 15** Let \( f : (B(r), g) \to Y \) be a harmonic map. For any \( r_0 \in (0, r) \), there exists \( \sigma_0 > 0 \) and \( \gamma > 0 \) dependent on the Lipschitz bound of \( g \), \( r_0 \) and the number of wedges of \( B(r) \) so that

\[
\sigma \mapsto \frac{E(\sigma)}{\sigma^{n-2+2\gamma}}, \quad \forall \sigma < \sigma_0
\]
is non-decreasing.

**Proof.** By [KS1], there exists \( Q_\sigma \) so that

\[
\int_{\partial B(\sigma)} d^2(f, Q_\sigma) d\Sigma_g = \inf_{Q \in Y \setminus B(\sigma)} \int_{\partial B(\sigma)} d^2(f, Q) d\Sigma_g.
\]
For the rest of the proof, we set
\[ I(\sigma) = \int_{\partial B(\sigma)} d^2(f, Q_\sigma) d\Sigma_g. \]

Thus, the Poincaré inequality (cf. Theorem 10) says there exists \( C_0 > 0 \) so that
\[ I(\sigma) \leq C_0 \sigma^2 \int_{\partial B(\sigma)} |\nabla_T f|^2 d\Sigma_g, \]

where \(|\nabla_T f|^2\) is the tangential part of \(|\nabla f|^2\). If we write
\[ h_{ij} = g(\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j}) \]

then
\[ |\nabla_T f|^2 = \frac{1}{r^2} h_{ij} \frac{\partial f}{\partial \theta_i} \frac{\partial f}{\partial \theta_j} = (1 + O(\sigma)) \left( |\nabla f|^2 - \left| \frac{\partial f}{\partial r} \right|^2 \right). \]

Thus,
\[ I(\sigma) \leq C_0 \sigma^2 (1 + O(\sigma)) \int_{\partial B(\sigma)} \left( |\nabla f|^2 - \left| \frac{\partial f}{\partial r} \right|^2 \right) d\Sigma_g. \tag{15} \]

Therefore,
\[
E^2(\sigma) \leq I(\sigma) \left( \int_{\partial B(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\Sigma_g \right)^\frac{1}{2} + c \sigma E'(\sigma)^\frac{1}{2} \leq 2 I(\sigma) \left( \int_{\partial B(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\Sigma_g + c^2 \sigma^2 E'(\sigma) \right) \leq C \sigma^2 \left( (n - 2 + O(\sigma)) E(\sigma) + \sigma(1 + O(\sigma)) \int_{\partial B(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\Sigma_g \right) \tag{11} \]

\[
\times \left( \int_{\partial B(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\Sigma_g + c^2 \sigma^2 E'(\sigma) \right) \leq C \sigma \left( (n - 2 + O(\sigma)) E(\sigma) + \sigma(1 + O(\sigma)) \int_{\partial B(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\Sigma_g \right) \tag{7} \]

\[
\times \left( (1 + O(\sigma)) \int_{\partial B(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\Sigma_g + c^2 \sigma (n - 2 + O(\sigma)) E(\sigma) \right) \leq C' \sigma^2 E^2(\sigma) + \sigma E(\sigma) \int_{\partial B(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\Sigma_g + \sigma^2 \left( \int_{\partial B(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\Sigma_g \right)^2 \]
\[ \leq C' \left( (\sigma^2 + \epsilon)E^2(\sigma) + \left( \frac{4}{\epsilon} + 1 \right) \sigma^2 \left( \int_{\partial B(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\Sigma_g \right)^2 \right). \]

Note that the constant $C$ and $C'$ depend only on the Lipschitz constant and the constant coming from the Poincaré inequality which only depends on the number of wedges of $B(r)$. Thus, the constants below also depend only on these quantities. By choosing $\sigma$ sufficiently small (depending on $C'$), we see that there exists a constant $K$ so that

\[ E(\sigma) \leq K\sigma \int_{\partial B(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\Sigma_g. \]

Using (7), we also have

\[ \sigma E'(\sigma) = (n - 2 + O(\sigma))E(\sigma) + (2\sigma + O(\sigma^2)) \int_{\partial B(\sigma)} \left| \frac{\partial f}{\partial r} \right|^2 d\Sigma_g \]

\[ \geq (n - 2 + O(\sigma))E(\sigma) + \frac{2 + O(\sigma)}{K} E(\sigma) \]

\[ = (n - 2 + \frac{2}{K} + O(\sigma))E(\sigma) \]

\[ \geq (n - 2 + 2\gamma)E(\sigma) \]

for $\gamma, \sigma > 0$ sufficiently small. This implies

\[ \frac{d}{d\sigma} \left( \log \frac{E(\sigma)}{\sigma^{n-2+2\gamma}} \right) \geq 0 \]

for $\sigma$ sufficiently small. Q.E.D.

**Lemma 16** For sufficiently small $\sigma$ depending on the Lipschitz bound of $g$ and any map $f : B \to Y$, we have

\[ \left| \frac{I'(\sigma)}{I(\sigma)} - \frac{n - 1}{\sigma} - \frac{1}{I(\sigma)} \int_{\partial B(\sigma)} \frac{\partial}{\partial r} d^2(f, Q) d\Sigma_g \right| \leq c_2 \]  \hspace{1cm} (16)

for some constant $c_2$ depending on the Lipschitz bound of $g$.

**Proof.** Let \( \{ \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_{n-1}} \} \) be the tangent basis corresponding to the polar coordinates \((r, \theta_1, ..., \theta_{n-1})\) on $W$. We define

\[ v(r, \theta) = \frac{1}{r^{n-1}} \sqrt{\left| \det \left( g \left( \frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j} \right) \right) \right|}. \]
By the fact that $g_{\alpha\beta}(0) = \delta_{\alpha\beta}$, we have
\[
|v(\sigma, \theta) - 1| = |v(\sigma, \theta) - \lim_{r \to 0} v(r, \theta)| \leq c' \sigma.
\]  
for some constant $c'$ depending on the Lipschitz bound of $g$. Since the measure induced on $\partial B(\sigma)$ by $g$ can be written $d\Sigma_g = \sigma^{n-1} v(\sigma, \theta) d\theta$, we have
\[
d\Sigma_g \geq \sigma^{n-1} (1 - c' \sigma) d\theta \geq \frac{1}{2} \sigma^{n-1} d\theta
\]
for sufficiently small $\sigma$. Thus,
\[
\frac{d}{d\sigma} \int_{\partial B(\sigma)} d^2(f, Q) d\Sigma_g
\]
\[
= \frac{d}{d\sigma} \int_{\partial B(\sigma)} d^2(f, Q) \sigma^{n-1} v(\sigma, \theta) d\theta
\]
\[
= \int_{\partial B(\sigma)} \frac{\partial}{\partial r} \left( d^2(f, Q) \sigma^{n-1} v(\sigma, \theta) \right) d\theta + \int_{\partial B(\sigma)} d^2(f, Q) (n-1) \sigma^{n-2} v(\sigma, \theta) d\theta
\]
\[
+ \int_{\partial B(\sigma)} d^2(f, Q) \sigma^{n-1} \frac{\partial v}{\partial r}(\sigma, \theta) d\theta
\]
\[
= \int_{\partial B(\sigma)} \frac{\partial}{\partial r} \left( d^2(f, Q) \right) d\Sigma_g + \frac{n-1}{\sigma} \int_{\partial B(\sigma)} d^2(f, Q) d\Sigma_g
\]
\[
+ \int_{\partial B(\sigma)} d^2(f, Q) \sigma^{n-1} \frac{\partial v}{\partial r}(\sigma, \theta) d\theta
\]
which in turn implies
\[
\left| \frac{I'(\sigma)}{I(\sigma)} - \frac{n-1}{\sigma} \right| = \frac{1}{I(\sigma)} \left| \int_{\partial B(\sigma)} \frac{\partial}{\partial r} d^2(f, Q) d\Sigma_g \right|
\]
\[
\leq \frac{c}{I(\sigma)} \int_{\partial B(\sigma)} d^2(f, Q) \sigma^{n-1} d\theta
\]
\[
\leq \frac{2c}{I(\sigma)} \int_{\partial B(\sigma)} d^2(f, Q) d\Sigma_g = 2c
\]
for sufficiently small $\sigma$. This immediately implies (16). Q.E.D.

Let $f : (B(r), g) \to Y$ be a harmonic map. Inequality (16) implies that there exists $\sigma_0$ sufficiently small so that for $\sigma < \sigma_0$,
\[
\frac{I'(\sigma)}{I(\sigma)} \leq \frac{n-1}{\sigma} + \frac{1}{I(\sigma)} \int_{\partial B(\sigma)} \frac{\partial}{\partial r} d^2(f, Q) d\Sigma_g + c_2.
\]  
(18)
Together, (8), (13) and (18) imply

\[
(1 + 3c\sigma) \frac{E'(\sigma)}{E(\sigma)} - \frac{I'(\sigma)}{I(\sigma)} + \frac{1}{\sigma} + c_3 \geq 0 \tag{19}
\]

where \(c_3 = c_1 + c_2\). We use these this inequality to prove a modified monotonicity which we describe below. For sake of simplicity of computation, we assume that \(3c \leq 1\) and \(\sigma_0 = 1\). These assumptions are without the loss of generality since we can rescale the metric \(g\) so that these conditions are met.

Let

\[ J(\sigma) = \max_{s \in [0, \sigma]} I(s) \]

and set

\[ A = \left\{ \sigma : \frac{E'(\sigma)}{E(\sigma)} - \frac{J'(\sigma)}{J(\sigma)} + \frac{1}{\sigma} + c_3 \leq 0 \right\}. \]

Since \(E'(\sigma) \geq 0\) and either \(J'(s) = 0\) or \(\frac{J'(s)}{J(s)} = \frac{I'(s)}{I(s)}\), we see that we have the following pair of inequalities:

\[
\frac{E'(\sigma)}{E(\sigma)} - \frac{J'(\sigma)}{J(\sigma)} + \frac{1}{\sigma} + c_3 \geq 0 \quad \text{for } \sigma \notin A \tag{20}
\]

\[
(1 + \sigma) \frac{E'(\sigma)}{E(\sigma)} - \frac{J'(\sigma)}{J(\sigma)} + \frac{1}{\sigma} + c_3 \geq 0 \quad \text{and } \sigma \in A. \tag{21}
\]

For \(\sigma \in (0, 1)\), set

\[
F(\sigma) = E(\sigma) \exp \left( - \int_{A \cap (\sigma, 1)} \frac{E'(s)}{E(s)} ds \right)
\]

**Theorem 17** For \(F(\sigma)\) defined above,

\[
\frac{F'(\sigma)}{F(\sigma)} = \begin{cases} 
\frac{E'(\sigma)}{E(\sigma)} & \text{for } \sigma \notin A \\
(1 + \sigma) \frac{E'(\sigma)}{E(\sigma)} & \text{for } \sigma \in A.
\end{cases} \tag{22}
\]

Consequently,

\[
\sigma \mapsto e^{c_3 \sigma} \frac{\sigma F(\sigma)}{J(\sigma)}
\]

is non-decreasing for \(\sigma\) sufficiently small.

**Remark.** If \(g\) is the Euclidean metric \(\delta\), then \(c = 0\) above and

\[
\sigma \mapsto \frac{\sigma E(\sigma)}{I(\sigma)}
\]

is non-decreasing.
Proof. If

\[ \varphi(\sigma) = -\int_{A \cap (\sigma, 1)} s \frac{E'(s)}{E(s)} ds, \]

then

\[ \varphi'(\sigma) = \lim_{\epsilon \to 0} \frac{\varphi(\sigma + \epsilon) - \varphi(\sigma - \epsilon)}{2 \epsilon} = \lim_{\epsilon \to 0} \frac{1}{2 \epsilon} \int_{A \cap [\sigma - \epsilon, \sigma + \epsilon]} s \frac{E'(s)}{E(s)} ds. \]

Therefore \( \varphi'(\sigma) = 0 \) for a.e. \( \sigma \not\in A \) and \( \varphi'(\sigma) = \sigma \frac{E'(\sigma)}{E(\sigma)} \) for a.e. \( \sigma \in A \). This implies (22) and together with (20) and (21), we obtain

\[ \frac{d}{ds} \log \frac{\sigma F'(\sigma)}{I(\sigma)} \geq \frac{F'(\sigma)}{F(\sigma)} - \frac{J'(\sigma)}{J(\sigma)} + \frac{1}{\sigma} \geq c_3. \]

which implies the monotonicity of \( \sigma \mapsto e^{c_3 \sigma} \frac{\sigma F'(\sigma)}{J(\sigma)} \). Q.E.D.

Theorem 17 shows the monotonicity involving a corrected energy term \( F(\sigma) \).

We now want to show that the correction factor is well-behaved as \( \sigma \to 0 \).

Lemma 18 There exists \( \epsilon_0 > 0 \) so that

\[ \epsilon_0 \leq \exp \left( -\int_{A \cap (\sigma, 1)} s \frac{E'(s)}{E(s)} ds \right). \]

Proof. It is sufficient to prove

\[ \int_{A \cap (0, 1)} s \frac{E'(s)}{E(s)} ds := \lim_{\tau \to 0} \int_{A \cap (\tau, 1)} s \frac{E'(s)}{E(s)} ds < \infty. \]

For \( s \in A \),

\[ s \frac{E'(s)}{E(s)} \leq s \frac{J'(s)}{J(s)} - 1 - c_3 s \leq s \frac{J'(s)}{J(s)}. \]  

by the definition of the set \( A \). Thus, it is sufficient to prove

\[ \lim_{\tau \to 0} \int_{A \cap (\tau, 1)} s \frac{J'(s)}{J(s)} ds < \infty. \]

We follow the argument of Proposition 3.1 in [M2].

Let \( M \) be sufficiently large so that for \( \sigma \in (0, 1] \),

\[ \int_{B(\sigma)} |\nabla f|^2 d\mu_g \leq ME(\sigma), \]

\[ N := e^{c_3} \frac{F(1)}{J(1)} \geq e^{c_3 \sigma} \frac{\sigma F'(\sigma)}{J(\sigma)} \geq \frac{\sigma F'(\sigma)}{J(\sigma)}. \]
and $K = MN$. Furthermore, let $0 < \theta_1 < \theta_2 \leq 1$ and $r_0 \in (\theta_1, \theta_2]$. For $s \in (\theta_1, r_0)$, we have by (16) that

\[
I'(s) \leq \int_{\partial B(s)} 2d(f, P) \frac{\partial}{\partial r} d(f, P) d\Sigma_g + \frac{n - 1 + c_2 s}{s} I(s)
\]

\[
\leq 2 \int_{\partial B(s)} d(f, P) |\nabla f| d\Sigma_g + \frac{n - 1 + c_2 s}{\theta_1} I(s)
\]

\[
\leq \int_{\partial B(s)} \left( \frac{1}{\epsilon} d^2(f, P) + \epsilon |\nabla f|^2 \right) d\Sigma_g + \frac{n - 1 + c_2 s}{\theta_1} I(s)
\]

\[
\leq \epsilon \int_{\partial B(s)} |\nabla f|^2 d\Sigma_g + \left( \frac{1}{\epsilon} + \frac{C}{\theta_1} \right) I(s)
\]

for some sufficiently large $C$. Therefore,

\[
I(r_0) - I(\theta_1) = \int_{\theta_1}^{r_0} I'(s) ds \leq \epsilon M E(r_0) + \left( \frac{1}{\epsilon} + \frac{C}{\theta_1} \right) \int_{\theta_1}^{r_0} I(s) ds.
\]

(24)

Hence

\[
I(r_0) - \epsilon M E(r_0) \leq I(\theta_1) + \left( \frac{1}{\epsilon} + \frac{C}{\theta_1} \right) (r_0 - \theta_1) \max_{s \in [\theta_1, r_0]} I(s).
\]

Since $r_0 \in (\theta_1, \theta_2]$ is arbitrary,

\[
\max_{s \in [\theta_1, \theta_2]} I(s) - \epsilon M E(\theta_2) \leq I(\theta_1) + \left( \frac{1}{\epsilon} + \frac{C}{\theta_1} \right) (\theta_2 - \theta_1) \max_{s \in [\theta_1, \theta_2]} I(s),
\]

which then implies

\[
\left[ 1 - \left( \frac{1}{\epsilon} + \frac{C}{\theta_1} \right) (\theta_2 - \theta_1) \right] \max_{s \in [\theta_1, \theta_2]} I(s) - \epsilon M E(\theta_2) \leq I(\theta_1).
\]

If $\max_{s \in [0, \theta_1]} I(s) \geq \max_{s \in [\theta_1, \theta_2]} I(s)$ then $J(\sigma)$ is identically equal to a constant in $[\theta_1, \theta_2]$. If $\max_{s \in [0, \theta_1]} I(s) \leq \max_{s \in [\theta_1, \theta_2]} I(s)$, then $\max_{s \in [\theta_1, \theta_2]} I(s) = \max_{s \in [\theta_1, \theta_2]} I(s)$. Either way, we have

\[
\left[ 1 - \left( \frac{1}{\epsilon} + \frac{C}{\theta_1} \right) (\theta_2 - \theta_1) \right] \max_{s \in [\theta_1, \theta_2]} J(s) - \epsilon M E(\theta_2) \leq J(\theta_1),
\]

which immediately implies

\[
\left[ 1 - \left( \frac{1}{\epsilon} + \frac{C}{\theta_1} \right) (\theta_2 - \theta_1) \right] J(\theta_2) - \epsilon M E(\theta_2) \leq J(\theta_1).
\]

For $\theta_0 \in (0, 1)$ to be determined later, we set

\[
\epsilon = \frac{\theta_0^{\frac{n-1}{n-\alpha_0}}}{2K}
\]
to obtain
\[
\left[ 1 - \left( 2K\theta_0^{-\frac{j(1-\theta_0^n)}{\theta_0^n}} + \frac{C}{\theta_0^n} \right) (\theta_2 - \theta_1) \right] J(\theta_2) - \frac{E(\theta_2)}{2N} \theta_0^{-\frac{j(1-\theta_0^n)}{\theta_0^n} + n} \leq J(\theta_1) \quad (25)
\]

Let
\[
\phi(\theta, n, j) = \frac{1}{2} - \left( 2K\theta_0^{-\frac{j(1-\theta_0^n)}{\theta_0^n}} + \frac{C}{\theta_0^n} \right) (1 - \theta).
\]

Then
\[
\lim_{\theta \to 1} \phi(\theta, n, j) = \frac{1}{2}
\]
uniformly independently of \(j, n\). Therefore, there exists \(\theta_0\) sufficiently close to 1 so that \(\phi(\theta_0, n, j) > \frac{1}{4}\) independently of \(n\) and \(j\). Choose \(j\) so that \(\theta_0^j < \frac{1}{4}\).

Then
\[
\frac{1}{2} - \left( 2K\theta_0^{-\frac{j(1-\theta_0^n)}{\theta_0^n}} + \frac{C}{\theta_0^n} \right) (1 - \theta_0^j) > \theta_0^j \quad (26)
\]
for any \(n\).

Since \(F(1) = E(1)\) by definition, we have that
\[
\frac{E(1)}{N} = \frac{J(1)}{e^c} \leq J(1).
\]
Thus, (25) with \(n = 0, \theta_1 = \theta_0\) and \(\theta_2 = 1\) implies
\[
\left[ \frac{1}{2} - \left( 2K\theta_0^{-\frac{j(1-\theta_0^n)}{\theta_0^n}} + \frac{C}{\theta_0^n} \right) (1 - \theta_0) \right] J(1) \leq J(\theta_0)
\]
and by inequality (26), \(\theta_0^j J(1) < J(\theta_0)\). Now suppose \(\theta_0^j J(\theta_0^k) < J(\theta_0^{k+1})\) for \(k = 1, ..., n - 1\). Then
\[
\int_{\theta_0^n}^{1} s \frac{d}{ds} \log J(s) ds = \sum_{k=0}^{n-1} \int_{\theta_0^{k+1}}^{\theta_0^k} s \frac{d}{ds} \log J(s) ds
\]
\[
\leq \sum_{k=0}^{n-1} \theta_0^k \int_{\theta_0^{k+1}}^{\theta_0^k} \frac{d}{ds} \log J(s) ds
\]
\[
\leq \sum_{k=0}^{n-1} \theta_0^k \log \frac{J(\theta_0^k)}{J(\theta_0^{k+1})}
\]
\[
\leq \sum_{k=0}^{n-1} \theta_0^k \log \theta_0^{-j} \theta_0^k \quad \text{(by the inductive hypothesis)}
\]
\[
= \sum_{k=0}^{n-1} \theta_0^{-j} \theta_0^k
\]
\[
\begin{align*}
&= \log \frac{1}{\theta_0} \sum_{k=\theta_0}^{\theta_0 - j} \theta_0^k \\
&= \log \theta_0^{- j} \sum_{k=\theta_0}^{\theta_0 - j} \theta_0^k
\end{align*}
\]

Using the fact that \( \frac{J'(s)}{J(s)} \geq 0 \), we obtain
\[
\begin{align*}
\log F(1) - F(\theta_0^n) &\leq \log \theta_0^{- j} \sum_{k=\theta_0}^{\theta_0 - j} \theta_0^k \\
&= \log \theta_0^{- j} \sum_{k=\theta_0}^{\theta_0 - j} \theta_0^k
\end{align*}
\]

Therefore, using the fact that \( E(1) = F(1) \) and the definition of \( N \), we obtain
\[
\begin{align*}
\frac{\theta_0^{j(1-\theta_0^n)}}{\theta_0^{1-\theta_0^n}} E(\theta_0^n) &\leq F(\theta_0^n) \leq \frac{N J(\theta_0^n)}{\theta_0^n}.
\end{align*}
\]

Thus, we can use the inequality
\[
\begin{align*}
\frac{E(\theta_0^n)}{2N} \theta_0^{j(1-\theta_0^n) + n} &\leq J(\theta_0^n)
\end{align*}
\]
in (25) with \( \theta_1 = \theta_0^{n+1} \), and \( \theta_2 = \theta_0^n \) to obtain
\[
\left[ \frac{1}{2} - \left( 2K \theta_0^{- j(1-\theta_0^n)} + C \theta_0 \right) (1 - \theta_0) \right] J(\theta_0^n) \leq J(\theta_0^{n+1})
\]
and by inequality (26),
\[
\theta_0^{j(1-\theta_0^n)} < J(\theta_0^{n+1}). \tag{27}
\]

By induction, inequality (27) holds for all \( n \) which in turn implies that
\[
\int_{\theta_0^n}^1 s \frac{d}{ds} \log J(s) ds \leq \log \theta_0^{- j(1-\theta_0^n)}
\]
holds for all \( n \). Letting \( n \to \infty \), we obtain
\[
\int_{\theta_0^{(0,1)}} s \frac{d}{ds} \log J(s) ds \leq \int_{0}^{1} s \frac{d}{ds} \log J(s) ds \leq \log \theta_0^{- j(1-\theta_0^n)} < \infty.
\]
Q.E.D.
Corollary 19 There exists a constant $\alpha < \infty$ so that

$$\alpha = \lim_{\sigma \to 0} \frac{\sigma E(\sigma)}{I(\sigma)}.$$  

We say $\alpha$ is the order of $f$ at 0.

Proof. By Lemma 18,

$$\lim_{\sigma \to 0} \frac{\sigma E(\sigma)}{I(\sigma)} = \lim_{\sigma \to 0} \left( \frac{\sigma F(\sigma)}{I(\sigma)} \cdot \frac{F(\sigma)}{E(\sigma)} \right) = \lim_{\sigma \to 0} e^{c_0 \sigma} \frac{\sigma F(\sigma)}{I(\sigma)} \cdot \exp \left( - \int_{A \cap (\sigma, 1)} s \frac{E'(s)}{E(s)} ds \right) < \infty.$$  

Q.E.D.

Lemma 20 Suppose $f : (B(r), g) \to (Y, d)$ is a harmonic map and

$$\alpha = \lim_{\sigma \to 0} \frac{\sigma E(\sigma)}{I(\sigma)}.$$  

There exists a constant $c_0$ and $\sigma_0$ depending only on the Lipschitz constant of $g$ so that if

$$\tilde{E}(\sigma) = E(\sigma) \exp \left( c_0 \int_{A \cap (0, \sigma)} s \frac{E'(s)}{E(s)} ds \right),$$

then

$$\sigma \mapsto e^{c_0 \sigma} \frac{\tilde{E}(\sigma)}{I(\sigma)}$$

and

$$\sigma \mapsto e^{c_0 \sigma} \frac{\tilde{E}(\sigma)}{\sigma^{2a+\nu-2}}$$

are non-decreasing functions for $\sigma \in (0, \sigma_0)$.

Proof. Set

$$G(\sigma) = E(\sigma) \exp \left( \int_{A \cap (0, \sigma)} s \frac{E'(s)}{E(s)} ds \right).$$

By definition,

$$\frac{G'(\sigma)}{G(\sigma)} = (1 + \sigma) \frac{E'(\sigma)}{E(\sigma)}.$$  

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for \( \sigma \in A \) and \( G'(\sigma) = E'(\sigma) \) for \( \sigma \notin A \). Thus, by (19), (recall we are assuming \( 3c \leq 1 \) for computational simplicity)

\[
0 \leq \frac{G'(\sigma)}{G(\sigma)} - \frac{I'(\sigma)}{I(\sigma)} + \frac{1}{\sigma} + c_3
\]

and

\[
\sigma \mapsto e^{c_3 \sigma} \frac{G(\sigma)}{I(\sigma)}
\]

is non-decreasing. By choosing \( c_0 \geq 1 \),

\[
\frac{G'(\sigma)}{G(\sigma)} \leq \frac{E'(\sigma)}{E(\sigma)}.
\]

and the first claim follows immediately. Since

\[
\lim_{\sigma \to 0} \frac{G(\sigma)}{E(\sigma)} = \lim_{\sigma \to 0} \exp\left( \int_{A \cap (0, \sigma)} \frac{E'(s)}{E(s)} \, ds \right),
\]

we have

\[
\alpha = \lim_{c \to 0} \frac{\sigma E(\sigma)}{I(\sigma)} = \lim_{c \to 0} \frac{\sigma G(\sigma)}{I(\sigma)} = \lim_{c \to 0} \frac{E(\sigma)}{I(\sigma)} = \lim_{c \to 0} e^{c_3 \sigma} \frac{G(\sigma)}{I(\sigma)}.
\]

Therefore, we see that

\[
\alpha \leq e^{c_3 \sigma} \frac{G(\sigma)}{I(\sigma)} = e^{c_3 \sigma} \frac{E(\sigma)}{I(\sigma)} \cdot \exp\left( \int_{0}^{\sigma} \frac{E'(s)}{E(s)} \, ds \right)
\]

(28)

and

\[
\frac{\sigma E(\sigma)}{I(\sigma)} \leq e^{c_3 \sigma} \frac{G(\sigma)}{I(\sigma)} \leq e^{c_3 \sigma} G(1) =: K.
\]

(29)

Now by the proof of Lemma 18, if \( \theta_{n+1} \leq \sigma < \theta_{n} \), then

\[
\int_{A \cap (0, \sigma)} \frac{E'(s)}{E(s)} \, ds \leq \int_{0}^{\sigma} \frac{J'(s)}{J(s)} \, ds \leq \int_{0}^{\theta_{n+1}} \frac{J'(s)}{J(s)} \, ds \leq \int_{0}^{1} \frac{J'(s)}{J(s)} \, ds - \int_{\theta_{n}}^{1} \frac{J'(s)}{J(s)} \, ds \leq \log \theta_{0} \frac{\theta_{n+1}}{\theta_{0} - \theta_{n}} - \log \theta_{0} \frac{j(1 - \theta_{n})}{1 - \theta_{n}} \leq c_4 \theta_{0} \sigma =: c_5 \sigma.
\]

(30)
Thus, this implies
\[ E(\sigma) \leq \bar{E}(\sigma) \leq e^{c_0 \sigma} E(\sigma), \]
and (28) implies
\[ \alpha \leq e^{(c_3 + c_5)\sigma} \frac{E(\sigma)}{I(\sigma)}. \] (31)

Furthermore, (12) and (16) implies that
\[
2E(\sigma) \leq \int_{\partial B(\sigma)} \frac{\partial}{\partial r} d^2(f, Q) d\Sigma_g + I(\sigma) + k\sigma^2 E'(\sigma)
\]
\[ \leq I'(\sigma) - \frac{n-1}{\sigma} I(\sigma) + I(\sigma) + k\sigma^2 E'(\sigma) \]
for some constant \( k \) depending on the Lipschitz constant of \( g \). The above two inequalities along with (29) imply
\[
\frac{2\alpha + n - 1 - O(\sigma)}{\sigma} \leq \frac{I'(\sigma)}{I(\sigma)} + k\sigma^2 \frac{E'(\sigma)}{E(\sigma)}
\]
\[ \leq \frac{I'(\sigma)}{I(\sigma)} + kK \sigma \frac{E'(\sigma)}{E(\sigma)} \]
\[ \leq \frac{G'(\sigma)}{G(\sigma)} + \frac{1}{\sigma} + kK \sigma \frac{E'(\sigma)}{E(\sigma)}. \] (32)

If \( c_0 \geq kK + 1 \), then
\[
\frac{E'(\sigma)}{E(\sigma)} = (1 + c_0 \sigma) \frac{E'(\sigma)}{E(\sigma)} \geq (1 + (kK + 1)\sigma) \frac{E'(\sigma)}{E(\sigma)} = \frac{G'(\sigma)}{G(\sigma)} + kK \sigma \frac{E'(\sigma)}{E(\sigma)}. \]

Therefore, using (32), we can choose \( c_0 \) sufficiently large so that
\[
\frac{2\alpha + n - 2}{\sigma} - c_0 \leq \frac{\bar{E}'(\sigma)}{\bar{E}(\sigma)}.
\]
Therefore,
\[
\sigma \mapsto e^{c_0 \sigma} \frac{\bar{E}(\sigma)}{\sigma^{2\alpha + n - 2}} \]
is non-decreasing. Q.E.D.

**Corollary 21** Let \( B \) be a dimension-\( n \), codimension-\( \nu \) local model, \( g \) an uniformly elliptic Riemannian Lipschitz metric defined on \( B(r) \), \((Y, d)\) a NPC space and \( f : (B(r), g) \to (Y, d) \) a harmonic map. Then there exist constants \( k \) and \( \sigma_0 \) depending on the Lipschitz bound of \( g \) so that for \( 0 < \sigma \leq \rho \leq \sigma_0 \),
\[
\frac{\sigma E(\sigma)}{I(\sigma)} \leq e^{k\rho} \frac{\bar{E}(\rho)}{I(\rho)}, \] (33)
\[
\frac{E(\sigma)}{\sigma^{2\alpha+n-2}} \leq e^{k\rho} \frac{E(\rho)}{\rho^{2\alpha+n-2}}.
\]

and

\[
\alpha \leq e^{k\sigma} \frac{E(\sigma)}{I(\sigma)}.
\]

**Proof.** Lemma 20 and (31) immediately imply (33) and (34). Inequality (35) is a restatement of (31). Q.E.D.

So far in this section, we assumed that our metric \(g\) is normalized at 0 (i.e. \(g_{\alpha\beta}(0) = \delta_{\alpha\beta}\)). We will relax this assumption and still show the monotonicity formulae for the energy of a harmonic map (cf. Proposition 22 below). Let \(B\) be a dimension-\(n\), codimension-(\(n-k\)) local model and \(g\) a Lipschitz metric on \(B\) with ellipticity constant \(\lambda \in (0, 1]\). For \(x \in B\), recall that \(R(x)\) is defined to be the radius of the largest homogeneous ball centered at \(x\) contained in \(B\). Assume that \(x\) is a codimension-(\(n-j\)) singular point. Let \(B'\) be a dimension-\(n\), codimension-(\(n-j\)) local model and \(L_x : B'(\lambda R(x)) \rightarrow L_x(B'(\lambda R(x))) \subset B\) be a homeomorphism satisfying properties (i) through (iv) of Proposition 4. In particular, recall this implies that \(h := L^*_x g\) is a normalized metric. If \(f : B \rightarrow Y\) is any finite energy map, then \(f \circ L_x\) is defined on \(B'_{x}(\lambda R(x))\). Moreover,

\[
\int_{B_x(\sigma)} |\nabla f|^2 d\mu_g = \int_{L_x^{-1}(B_x(\sigma))} |\nabla (f \circ L_x)|^2 d\mu_h
\]

and

\[
\int_{B'(\sigma)} |\nabla (f \circ L_x)|^2 d\mu_h = \int_{L_x(B'(\sigma))} |\nabla f|^2 d\mu_g.
\]

This in turn implies that if \(f\) is a harmonic map with respect to the metric \(g\), then \(f \circ L_x\) is a harmonic map with respect to the metric \(h\). We call \(f \circ L_x\) the normalized harmonic map at \(x\). Recall that \(\sigma_0\) was defined in the previous section as the upper bound for which monotonicity formulae of Lemma 15 and Corollary 21 are valid for any harmonic map from a local model with a normalized metric. Therefore, these monotonicity formulae for \(f \circ L_x\) are valid for balls \(B'(\sigma)\) contained in \(B'(\lambda R(x))\), where

\[
r_0(x) := \min\{\sigma_0, R(x)\}.
\]

For a harmonic map \(f\), we define the order of \(f\) at \(x\) as

\[
\alpha_x = \lim_{\sigma \to 0} \frac{\sigma \int_{B'(\sigma)} |\nabla (f \circ L_x)|^2 d\mu_h}{\int_{B'(\sigma)} d^2(f \circ L_x, f \circ L_x(0)) d\Sigma_h}.
\]

The above limit exists by Corollary 19. We define \(E_x(\sigma)\) and \(I_x(\sigma)\) for \(\sigma\) sufficiently small by setting

\[
E_x(\sigma) = \int_{B_x(\sigma)} |\nabla f|^2 d\mu_h
\]
\[ I_x(\sigma) = \int_{\partial B_x(\sigma)} d^2(f, f(0)) d\Sigma_h. \]

**Proposition 22** Let \( B \) be a dimension-\( n \), codimension-\( (n-k) \) local model and \( g \) a Lipschitz metric \( g \) defined on \( B \) with ellipticity constant \( \lambda \). Then there exist constants \( \gamma \) and \( C \) dependent on the Lipschitz bound and the ellipticity constant of \( g \) so that for every \( x \in B \),

\[
\frac{E_x(\sigma)}{\sigma^{n-2+2\gamma}} \leq C \frac{E_x(\rho)}{\rho^{n-2+2\gamma}}, \quad 0 < \sigma < \rho \leq r(x) \tag{39}
\]

and

\[
\frac{E_x(\sigma)}{\sigma^{n-2+2\alpha_x}} \leq C \frac{E_x(\rho)}{\rho^{n-2+2\alpha_x}}, \quad 0 < \sigma < \rho \leq r(x) \tag{40}
\]

where

\[ r(x) = \lambda r_0(x). \tag{41} \]

**Proof.** Let \( L_x \) as above and set

\[ \mathcal{E}(\sigma) = \int_{B^x(\sigma)} |\nabla(f \circ L_x)|^2 d\mu_h. \]

Lemma 15 and Corollary 21 imply that there exists a constant \( c \geq 1 \) so that

\[
\frac{\mathcal{E}(s)}{s^{n-2+2}\Gamma} \leq c \frac{\mathcal{E}(r)}{r^{n-2+2}\Gamma}, \quad 0 < s < r \leq r_0(x) \tag{42}
\]

and

\[
\frac{\mathcal{E}(s)}{s^{n-2+2\alpha_x}} \leq c \frac{\mathcal{E}(r)}{r^{n-2+2\alpha_x}}, \quad 0 < s < r \leq r_0(x) \tag{43}
\]

with \( r_0(x) \) as in (38). Let \( \Gamma = n - 2 + 2\gamma \) or \( \Gamma = n - 2 + 2\alpha_x \). Fix \( \sigma, \rho \) so that \( 0 < \sigma < \rho \leq r(x) \). Then, since \( \lambda \leq 1 \), \( 0 < \lambda^{-1} \sigma, \lambda \rho \leq r_0(x) \). We prove (39) and (40) by considering the following two cases. In the first case, we assume \( \lambda^{-1} \sigma \leq \lambda \rho \). We then have

\[
\frac{E_x(\sigma)}{\sigma^{\Gamma}} \leq \frac{\mathcal{E}(\lambda^{-1} \sigma)}{\sigma^{\Gamma}} \text{ by (36) and the fact that } L_x^{-1}(B_x(\sigma)) \subset B'(\lambda^{-1} \sigma)
\]

\[
\leq \frac{1}{\lambda^{\Gamma}} \frac{\mathcal{E}(\lambda^{-1} \sigma)}{(\lambda^{-1} \sigma)^{\Gamma}}
\]

\[
\leq \frac{c}{\lambda^{\Gamma}} \frac{\mathcal{E}(\lambda \rho)}{(\lambda \rho)^{\Gamma}} \text{ by (42) or (43) and the assumption that } \lambda^{-1} \sigma \leq \lambda \rho
\]

\[
\leq \frac{c}{\lambda^{\Gamma}} \frac{E_x(\rho)}{(\lambda \rho)^{\Gamma}} \text{ by (37) and the fact that } L_x(B'(\lambda \rho)) \subset B_x(\rho)
\]

\[
\leq \frac{c}{\lambda^{\Gamma}} \frac{E_x(\rho)}{\rho^{\Gamma}}.
\]
In the second case, we assume $\lambda^{-1}\sigma > \lambda \rho$. We then have
\[
\frac{E_x(\sigma)}{\sigma^4} \leq \frac{E_x(\rho)}{\rho^4} \leq \frac{E_x(\rho)}{(\lambda^2 \rho)^4} \leq \frac{1}{\lambda^4} \frac{E_x(\rho)}{\rho^4}.
\]
In either case, we have proven our assertion by setting $C = \frac{\sigma}{\lambda^4}$. Q.E.D.

4 Hölder continuity

In this section, we prove the Hölder continuity of a harmonic map from a Riemannian complex into a NPC space $Y$. Such a result in the case when the domain metric is smooth was discussed in [EF] and [Ch]. Using the results of the previous section, we are able to consider a Lipschitz metric $g$. Moreover, we provide the explicit dependence of the Hölder exponent and Hölder constant on $g$, $E^f$ and the geometry of $B$. In the later sections, we give a condition for which the Hölder continuity can be improved to Lipschitz continuity. Our proof follows the approach in [GS] and [Ch]. The main technical difficulty is that monotonicity only works for small balls (cf. Remark 3.3 [DM1]). Therefore in order to obtain the energy decay estimate for large balls (cf. Proposition 27), we need the rather technical inductive process described in Proposition 25 and Corollary 26. We first prove some results pertaining to the geometry of local models.

**Proposition 23** Fix integers $k, n$ so that $0 \leq k < n$. Assume that the sets $B_k, \ldots, B_n$ have the following properties:

(1) For each $j \in \{k, k+1, \ldots, n\}$, $B_j$ is a finite set of dimension-$n$, codimension-$(n-j)$ local models, and

(2) if $B \in B_j$ for some $j \in \{k, k+1, \ldots, n-1\}$, then for any $x \in B$ and $\sigma \leq R(x)$, we have $B_x(\sigma)$ is isometric $B'(\sigma)$ where $B' \in B_i$ for some $i \in \{j, j+1, \ldots, n\}$.

Then all $j \in \{k, \ldots, n\}$ and $B \in B_j$, there exists $\kappa(B) \geq 1$ so that for all $i \in \{j, \ldots, n\}$,
\[
\frac{|x - \bar{x}|}{R(x)} < \kappa(B), \quad \forall x \in S_i - S_{i-1} \subset B, \forall \bar{x} \in \pi_{i-1}(x).
\]

**Proof.** We first make the following observation. Let $B$ be any dimension-$n$, codimension-$(n-k)$ local model and $x \in B$. Recall that $D = S_k$ is isometric
to \( \mathbb{R}^k \) and hence the closest point projection map \( \pi_D : B \to D \) is well-defined. For any \( x \in B - D \), let \( t \mapsto x_t \) be the constant speed parameterization of a ray starting from \( \pi_D(x) \) and going through \( x = x_1 \). Assume \( x \in S_i - S_{i-1} \) and let \( \bar{x} \in \pi_{i-1}(x) \). Since \( t \mapsto x_t \) and \( t \mapsto \pi_t \) are rays from \( \pi_D(\pi) = \pi_D(x) \), we see that \( \pi_{i} \in \pi_{i-1}(x_t) \) and \( t|x - \pi| = |x_t - \pi_t| \). Furthermore, we also see that \( tR(x) = R(x_t) \). Thus, we observe that

\[
\frac{|x - \bar{x}|}{R(x)} = \frac{|x_t - \pi_t|}{R(x_t)}, \quad \forall t \in (0, \infty).
\]  \hspace{1cm} (44)

We now proceed with the proof of the assertion by reverse induction on \( j \). First note that there is nothing to prove for \( j = n \) since \( B_n = \{ \mathbb{R}^n \} \). Assume that the assertion is true for \( m = n, n-1, ..., j+1 \). By (44), we only need to show that for each \( B \in \mathcal{B}_j \), there exists \( \kappa(B) \) so that for any \( i \in \{ j, ..., n \} \),

\[
\frac{|x - \bar{x}|}{R(x)} < \kappa(B), \quad \forall x \in U \cap (S_i - S_{i-1}) \subset B, \forall \bar{x} \in \pi_{i-1}(x)
\]

where \( U \) is the set of points of \( B \) at a distance 1 from \( D \). Suppose this is not true, i.e. for some fixed \( i \), there exists \( B \in \mathcal{B}_j \) and a sequence \( y_n \in U \cap (S_i - S_{i-1}) \) so that

\[
\frac{|y_n - \bar{y}_n|}{R(y_n)} \to \infty
\]  \hspace{1cm} (45)

with \( \bar{y}_n \in \pi_{i-1}(y_n) \). Since \( U \) is a compact set, we may assume (by choosing a subsequence if necessary) that \( y_n \to y \). By the definition of \( U \), \( y \in S_m - S_{m-1} \) for \( m > j \). Consider a dilation of \( B \) centered at \( y \) by a factor of \( \frac{1}{|y_n - y|} \) and let \( z_n \) and \( \bar{z}_n \) be the images of \( y_n \) and \( \bar{y}_n \) respectively under this dilation. (By definition, the distance between \( z_n \) and \( y \) is normalized to be to 1.) Since dilation preserves projections, we see that \( \bar{z}_n \in \pi_{i-1}(z_n) \). Furthermore, since dilation also preserves ratio of distances,

\[
\frac{|z_n - \bar{z}_n|}{R(z_n)} = \frac{|y_n - \bar{y}_n|}{R(y_n)}.
\]  \hspace{1cm} (46)

By the facts that \( y \in S_m - S_{m-1} \), \( |y_n - y| \to 0 \) and assumption (2), we can assume that \( z_n \) and \( \bar{z}_n \) are points in the local model \( B' \in \mathcal{B}_m \). Since \( m > j \), we can apply the induction hypothesis and conclude that (46) is bounded which contradicts (45). This finishes the inductive step and the proof. Q.E.D.

**Corollary 24** Let \( B \) be a dimension-\( n \), codimension-(\( n - k \)) local model. There exists \( \kappa \geq 1 \) so that for any \( i \in \{ k, ..., n \} \),

\[
\frac{|x - \pi_{i}(x)|}{R(x)} < \kappa, \quad \forall x \in S_i - S_{i-1} \subset B.
\]
Proof. Apply Proposition 25 with $B_j$ for $j = k, \ldots, n$ defined to be the set of model spaces so that $B' \in B_j$ if and only if there exists $x \in S_j - S_{j-1}$ so that $B_x(R(x))$ is isometric to $B'(R(x))$. We are done by setting $\kappa = \kappa(B')$ where $B'$ is the unique element in $B_k$. Q.E.D.

Proposition 25 Fix integers $k, n$ so that $0 \leq k < n$. Assume that the sets $B_k, \ldots, B_n$ have the following properties:

(1) For each $j \in \{k, k+1, \ldots, n\}$, $B_j$ is a finite set of dimension-$n$, codimension-$(n-j)$ local models, and

(2) if $B \in B_j$ for some $j \in \{k, k+1, \ldots, n-1\}$, then for any $x \in B$ and $\sigma \leq R(x)$, we have $B_x(\sigma)$ is isometric $B'(\sigma)$ where $B' \in B_i$ for some $i \in \{j, j+1, \ldots, n\}$.

There exists $C > 0$ so that for any $B \in B_j$, $j \in \{k, k+1, \ldots, n\}$, and $x \in B$, there exists an ordered sequence $x_1 \triangleright x_2 \triangleright \cdots \triangleright x_m$ of points in $B$ with $x_1 = x$, $x_m \in S_j$ and positive numbers $\sigma_1, \ldots, \sigma_{m-1}$ with the property that

$$\frac{\sigma_i}{R(x_i)} \leq C, \quad \frac{\sigma_j}{R(x_{i+1})} \leq 1 \text{ and } B_x(R(x_i)) \subset B_{x_{i+1}}(\sigma_i). \quad (47)$$

Proof. We first need some preliminary constructions on each element $B$ of $\bigcup_{j=k}^n B_j$. So fix $j$ and $B \in B_j$. Let $U$ be the set of points of $B$ at a distance 1 from $D = S_j$. Set

$$U_n = U, \quad U_{n-1} = \pi_{n-1}(U_n), \ldots, \quad U_j = \pi_j(U_{j+1}).$$

By the convexity of the faces of $B$, we see that $U_{j+1} \subset B - S_j$.

For $i = j+1, \ldots, n-1$, we define a positive number $R_i$ and a subset $N_i$ of $B$ by an inductive procedure.

• First we define $R_{j+1}$ and $N_{j+1}$.

Let $V_{j+1}$ be so that

$$U_{j+1} \subset V_{j+1} \subset S_{j+1} - S_j.$$ 

Thus, there exists $R_{j+1} > 0$ so that

$$R(x') \geq R_{j+1}, \quad \forall x' \in V_{j+1}.$$ 

We can choose a neighborhood $N_{j+1} \subset B$ of $U_{j+1}$ so that for every $x \in N_{j+1}$ and $x' \in \Pi_{j+1}(x)$ with $\sigma' = |x - x'|$, we have that
\[ B_{x'}(\sigma') \subset N_{j+1} \]
\[ \Pi_{j+1}(N_{j+1}) \subset V_{j+1} \]
and
\[ U_{j+1} \subset N_{j+1} \subset \{ x \in B - S_j : 2\delta(x) < R_{j+1} \} \]
with \( \delta(x) \) the distance of \( x \) to \( S_j \). Thus,
\[ U_{j+1} \subset N_{j+1} \subset \{ x \in B - S_j : 2\delta(x) < R(x') \text{ where } x' \in \Pi_{j+1}(x) \} \].

- Assuming we have chosen positive numbers \( R_{j+1}, \ldots, R_{i-1} \) and open sets \( N_{j+1}, \ldots, N_{i-1} \) so that for \( l \in \{ j + 1, \ldots, i - 1 \} \),
\[ B_{x'}(\sigma') \subset N_l, \forall x' \in \Pi_l(x) \text{ where } x \in N_l \text{ and } \sigma' = |x - x'|, \]
\[ \Pi_l(N_l) \subset V_l \]
and
\[ U_l - \bigcup_{m=j+1}^{i-1} N_m \subset \subset N_l \subset \subset \{ x \in B - S_{l-1} : 2\lambda \delta(x) < R_l \text{ where } x' \in \Pi_l(x) \} \],
we define \( R_i \) and \( N_i \) as follows:

First note that
\[ U_i - \bigcup_{m=j+1}^{i-1} N_m \subset \subset U_i - S_{i-1}. \]
Thus, we can choose \( V_i \subset S_i \) be so that
\[ U_i - \bigcup_{m=j+1}^{i-1} N_m \subset \subset V_i \subset \subset U_i - S_{i-1}. \]
Thus, there exists \( R_i > 0 \) so that
\[ R(x') \geq R_i, \forall x' \in V_i. \]
We can choose a neighborhood \( N_i \subset B \) of \( U_i - \bigcup_{m=j+1}^{i-1} N_i \) so that for every \( x \in N_i \) and \( x' \in \Pi_i(x) \) with \( \sigma' = |x - x'| \), we have that
\[ B_{x'}(\sigma') \subset N_i \]
\[ \Pi_i(N_i) \subset V_i \]
and
\[ U_i - \bigcup_{m=j+1}^{i-1} N_i \subset \subset N_i \subset \subset \{ x \in B - S_{i-1} : 2\lambda \delta(x) < R_i \} \]
with \( \delta(x) \) the distance of \( x \) to \( S_i \). Thus,
\[ U_i - \bigcup_{m=j+1}^{i-1} N_i \subset \subset N_i \subset \subset \{ x \in B - S_{i-1} : 2\delta(x) < R(x') \text{ where } x' \in \Pi_i(x) \}. \]
In summary, we have constructed sets \( U = U_n, \ldots, U_{j+1} \), positive numbers \( R_{j+1}, \ldots, R_{n-1} \) and open sets \( N_{j+1}, \ldots, N_{n-1} \) so that for each \( i = j + 1, \ldots, n - 1 \), we have
\[
B_i(x') \subset N_i
\]
and
\[
\Pi_i(N_i) \subset V_i
\]
and
\[
U_1 = \bigcup_{i=j+1}^{n-1} N_l \subset N_i \subset \{ x \in B - S_i : 2\delta(x) < R_i \leq R(x') \text{ where } x' \in \Pi_i(x) \}.
\]
Since \( N = \bigcup_{i=j+1}^{n-1} N_l \) covers the singular set of \( U \), there exists \( R_n > 0 \) so that \( R(x) \geq R_n, \forall x \in U - N \).

In the above, for each \( j \in \{ k, \ldots, n \} \) and \( B \in \mathcal{B}_j \), we associated sets \( U = U(B), U_{n-1} = U_{n-1}(B), \ldots, U_{j+1} = U_{j+1}(B) \), positive numbers \( R_{j+1} = R_{j+1}(B), \ldots, R_n = R_n(B) \) and open sets \( N_{j+1} = N_{j+1}(B), \ldots, N_{n-1} = N_{n-1}(B), N = N(B) \). Let
\[
C(B) = \max\left\{ \frac{2}{R_{j+1}(B)}, \ldots, \frac{2}{R_{n}(B)} \right\},
\]
and
\[
C = \max\{ C(B) : B \in \mathcal{B}_j, j = k, \ldots, n \}.
\]
Furthermore, \( l = k, \ldots, n \), let
\[
\hat{R}_l = \min\{ R_l(B) : B \in \mathcal{B}_j, j = k, \ldots, n \}.
\]
We now proceed with the proof of the proposition. Since \( \mathcal{B}_n = \{ \mathbb{R}^n \} \), there is nothing to prove for \( \mathcal{B}_n \). We now prove the assertion for \( \mathcal{B}_j \) for any \( j = k, \ldots, n \) by doing a reverse induction; more specifically, assume that the assertion is true for \( \mathcal{B}_n, \mathcal{B}_{n-1}, \ldots, \mathcal{B}_{j+1} \) and prove the assertion for \( \mathcal{B}_j \).

Now given \( x \in B \in \mathcal{B}_j \), we need to show that there exists an ordered sequence
\[
x_1 \triangleright x_2 \triangleright \ldots \triangleright x_m
\]
of points in \( B \) with \( x_1 = x, x_m \in S_j \) and positive numbers \( \sigma_1, \ldots, \sigma_{m-1} \) satisfying (47).

If \( x \) is in the lowest dimensional stratum \( D = S_j \), there is nothing to prove so assume \( x \in B - S_j \). By the scale invariance of the assertion, we may assume that \( x \in U(B) \). If \( x \in U(B) - N(B) \), then let \( x_1 = x, x_2 \in \Pi_j(x_1) \). Since \( R(x_2) = \infty \), there is nothing to prove.

So assume \( x \in N(B) \); in particular, \( x \in N_l(B) \) for some \( l = j + 1, \ldots, n - 1 \). In this case, we use the inductive hypothesis. More specifically, choose \( x' \in \Pi_l(x) \) and let \( \sigma' = |x - x'| \). By assumption 2, we can identify \( B_{x'}(\sigma') \) with \( B'(\sigma') \) where \( B' \in \mathcal{B}_l \). The inductive hypothesis implies that for any \( B' \in \mathcal{B}_l \) any \( x \in B' \), there exists a sequence
\[
x_1 = x \triangleright x_2 \triangleright \ldots \triangleright x_m \in S_l
\]
and \( \sigma_1, \ldots, \sigma_{m-1} \) with the property that
\[
\frac{\sigma_i}{R(x_i)} \leq C, \quad \frac{\sigma_i}{R(x_{i+1})} \leq 1
\]
and
\[
B_{x_i}(R(x_i)) \subset B_{x_{i+1}}(\sigma_i).
\]
Since \( x \in U(B) \), we have that \( x_m \in U_l(B) = \Pi_l(U(B)) \). Thus, \( R(x_m) \geq R_l(B) \geq \hat{R} \). Therefore, if we set \( \sigma_m = 2 \),
\[
\frac{\sigma_m}{R(x_{m+1})} \leq 1
\]
since \( R(x_{m+1}) = \infty \) and
\[
\frac{\sigma_m}{R(x_m)} = \frac{2}{R(x_m)} \leq \frac{2}{\hat{R}} \leq C
\]
Furthermore, since the distance of \( x \) to \( S_j \) is equal to 1, \( R(x_m) \leq 1 \). Hence
\[
B_{x_m}(R(x_m)) \subset B_{x_{m+1}}(\sigma_m).
\]
This completes the inductive step and finishes the proof of Proposition 25.

\[\text{q.e.d.}\]

**Corollary 26** Let \( B \) be a dimension-\( n \), codimension-(\( n - k \)) local model, \( g \) an uniformly elliptic Lipschitz metric defined on \( B(r) \), \( r(x) \) as defined in (41), \( (Y, d) \) a NPC space and \( f : (B(r), g) \to (Y, d) \) a finite energy map. Then there exist \( C > 0 \) and \( R > 0 \) depending on \( g \) and the geometry of \( B \) so that for every \( x \in B(\frac{r}{2}) \), there exist a sequence of points
\[
x = x_1 \triangleright \ldots \triangleright x_m
\]
and a sequence of positive numbers
\[
\sigma_1, \ldots, \sigma_{m-1}
\]
so that for \( i = 1, \ldots, m - 1 \),
\[
\frac{\sigma_i}{r(x_i)} \leq C, \quad \frac{\sigma_i}{r(x_{i+1})} \leq 1, \quad E_{x_i}(r(x_i)) \leq E_{x_{i+1}}(\sigma_i)
\]
and
\[
r(x_m) \geq R_0.
\]
Proof. We define $B_j$ for $j = k, \ldots, n$ to be the set of model spaces so that $B' \in B_j$ if and only if there exists $x \in S_j - S_{j-1}$ so that $B_x(R(x))$ is isometric to $B'(R(x))$. Then $B_k, \ldots, B_n$ satisfy conditions (1) and (2) of Proposition 25. For each $y \in B(\frac{r}{2})$, assume $y \in S_j - S_{j-1}$. Choose $r_y > 0$ sufficiently small so that $B_y(r_y)$ is compactly supported away from $S_{j-1}$ and $r_y < \sigma_0$. In this way, we see that $\lambda^2 R(y') = r(y')$ for $y' \in B_y(r_y)$ and $R_y := \inf \{ r(y') : y' \in B_y(r_y) \cap S_j \} > 0$. We can choose a finite covering $\{ B_y(r_y) : l = 1, \ldots, N \}$ of $B(\frac{r}{2})$. Given $x \in B(\frac{r}{2})$, there exists $l \in \{ 1, \ldots, N \}$ so that $x \in B_y(r_y)$. The point $y_l$ is an element of $S_j$ for some $j \in \{ k, \ldots, n \}$ and hence $B_{y_l}(r_{y_l})$ is isometric to $B \in B_j$. Applying Proposition 25 and noting that $r = \lambda^2 R$, we obtain sequences $x_1 \triangleright \ldots \triangleright x_m$ and $\sigma_1, \ldots, \sigma_{m-1}$ which satisfy

$$\frac{\sigma_i}{r(x_i)} \leq \lambda^{-2} C, \quad \frac{\sigma_i}{r(x_{i+1})} \leq \lambda^{-2}$$ and $B_{x_i}(\lambda^{-2} r(x_i)) \subset B_{x_{i+1}}(\sigma_i)$ \hfill (50)

by (47). By an abuse of notation, we will set $\sigma_i$ equal to $\lambda^2 \sigma_i$ to obtain

$$\frac{\sigma_i}{r(x_i)} \leq C, \quad \frac{\sigma_i}{r(x_{i+1})} \leq 1$$ and $B_{x_i}(r(x_i)) \subset B_{x_{i+1}}(\sigma_i)$ \hfill (51)

The inclusion above shows that

$$E_{x_i}(r(x_i)) \leq E_{x_{i+1}}(\sigma_i).$$

Finally, if we set $R_0 = \min \{ R_{y_1}, \ldots, R_{y_N} \}$, then we obtain $r(x_m) \geq R > 0$. Q.E.D.

Proposition 27 Let $B$ be a dimension-$n$, codimension-$(n - k)$ local model, $g$ an uniformly elliptic Lipschitz metric defined on $B(r)$, $r(x)$ as defined in (41), $(Y, d)$ a NPC space and $f : (B(r), g) \to (Y, d)$ a finite energy map. For $x \in B(\frac{r}{2})$, suppose there exist $\beta$ and $C$ so that

$$\frac{E_x(\sigma)}{\sigma^2 + 2 \beta} \leq C \frac{E_x(\rho)}{\rho^2 + 2 \beta}, \quad \text{for } 0 < \sigma \leq \rho \leq r(x).$$ \hfill (52)

Then there exist $K$ and $R > 0$ depending only on $E_f$, the ellipticity constant, the Lipschitz bound of $g$ and the geometry of $B(r)$ so that

$$E_x(\sigma) \leq K^2 \sigma^{2n - 2 + 2\beta}, \quad \forall x \in B(r/2), \sigma < R.$$ \hfill (53)

Proof. For the sake of simplicity, we will use $C$ as a generic constant in this proof. Set

$$R = \min \left\{ R_0, \frac{r}{2^{n-k+1}}, \sigma_0 \right\}$$

where $R_0$ is as in Corollary 26 and $\sigma_0$ as in Corollary 20. Let $x \in B(\frac{r}{2})$ and $\sigma < R$.

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Case 1. Assume that $\sigma \leq r(x)$. By (52),
\[
\frac{E_x(\sigma)}{\sigma^{2n-2+2\beta}} \leq C \frac{E_x(r(x))}{r(x)^{2n-2+2\beta}}.
\]
Let $x = x_1 \triangleright \ldots \triangleright x_m$ and $\sigma_1, \ldots, \sigma_{m-1}$ be as in Corollary 26. By (48) and (52),
\[
\frac{E_x(r(x_i))}{r(x_i)^{2n-2+2\beta}} \leq C \frac{E_x(\sigma_i)}{\sigma_i^{2n-2+2\beta}} \leq C \frac{E_x(r(x_{i+1}))}{r(x_{i+1})^{2n-2+2\beta}}
\]
for $i = 1, \ldots, m-1$. Additionally, by (49),
\[
\frac{E_x(r(x_m))}{r(x_m)^{2n-2+2\beta}} \leq \frac{E_x(r(x_m))}{R^{2n-2+2\beta}} \leq \frac{E'_r}{R^{2n-2+2\beta}}
\]
and hence
\[
\frac{E_x(\sigma)}{\sigma^{2n-2+2\beta}} \leq C \frac{E'_r}{R^{2n-2+2\beta}}.
\]
Set
\[
K_0 = \sqrt{\frac{E'_r}{R^{2n-2+2\beta}}}
\]
and we obtain
\[
E_x(\sigma) \leq K_0^2 \sigma^{2n-2+2\beta} \text{ whenever } \sigma < r(x).
\]

Case 2. Alternately, assume $r(x) < \sigma$. Set
\[
K = \left( \frac{2n^{n} \kappa^{n}}{\lambda^2 \sigma_0} \right)^{n-1+\beta} \max\{K_0, \sqrt{E'_r}\}
\]
where $\kappa \geq 1$ is defined in Corollary 24. Since $r(x) = \min\{\lambda \sigma_0, \lambda^2 R(x)\}$, either $\lambda \sigma_0 < \sigma$ or $R(x) < \frac{\sigma}{\lambda^2}$. First consider the case when $\lambda \sigma_0 < \sigma$. Then
\[
E_x(\sigma) \leq E_x(\sigma) \left( \frac{\sigma}{\lambda \sigma_0} \right)^{2n-2+2\beta} \leq \frac{E'_r}{(\lambda \sigma_0)^{2n-2+2\beta}} \sigma^{2n-2+2\beta} \leq K^2 \sigma^{2n-2+2\beta}.
\]
Next assume $R(x) < \frac{\sigma}{\lambda^2}$. The fact that $R(x') = \infty$ for every $x' \in D = S_k$ implies that $x \notin D = S_k$. So let $i \in \{k+1, \ldots, n\}$ be so that $x \in S_i - S_{i-1}$. Furthermore, let $x = y_1, \ldots, y_k \in D = S_k$ where $y_i \in \pi_{i-1}(y_i)$. We now follow the following finite step procedure.
Step 1. Since $R(y_i) < \frac{\sigma}{\lambda^2}$, there exists $l_1 \in \{k+1, \ldots, i-1\}$ so that
\[ R(y_m) < \frac{2^{i-m}\sigma}{\lambda^2}, \forall m = l_1 + 1, \ldots, i. \]

Thus, we can apply Corollary 24 to obtain
\[ |y_m - y_{m-1}| < \kappa R(y_m) < \frac{2^{i-m}\kappa}{\lambda^2} \sigma. \]

This implies
\[ B_{y_i} \left( \frac{\kappa}{\lambda^2} \sigma \right) \subset B_{y_{i-1}} \left( \frac{2\kappa}{\lambda^2} \sigma \right) \subset \ldots \subset B_{y_1} \left( \frac{2^{i-l_1}\kappa}{\lambda^2} \sigma \right). \]

Since $\frac{\kappa}{\lambda^2} \leq 1$, we also have
\[ B_{y_i} \left( \sigma \right) \subset B_{y_i} \left( \frac{\kappa}{\lambda^2} \sigma \right). \]

The above inclusions imply
\[ E_x(\sigma) \leq E_{y_1} \left( \frac{2^{i-l_1}\kappa}{\lambda^2} \sigma \right). \tag{53} \]

We now consider the following two cases:

(A) $\frac{2^{i-l_1}\kappa}{\lambda^2} \sigma \leq R(y_1)$ or (B) $R(y_1) < \frac{2^{i-l_1}\kappa}{\lambda^2} \sigma$.

If (A) is true, either
\[ \sigma_0 \leq \frac{2^{i-l_1}\kappa}{\lambda^2} \sigma \text{ or } \frac{2^{i-l_1}\kappa}{\lambda^2} \sigma < r(y_1). \]

In the former case, we use (53) and the fact that $1 \leq \frac{2^{i-l_1}\kappa}{\lambda^2} \sigma_0$ to see that
\[ E_x(\sigma) \leq E_{y_1} \left( \frac{2^{i-l_1}\kappa}{\lambda^2} \sigma \right) \cdot \left( \frac{2^{i-l_1}\kappa}{\lambda^2\sigma_0} \right)^{2n-2+2\beta} \sigma^{2n-2+2\beta} \leq K^2 \sigma^{2n-2+2\beta}. \]

In the latter case, we use (53) and Case 1 to see that
\[ E_x(\sigma) \leq E_{y_1} \left( \frac{2^{i-l_1}\kappa}{\lambda^2} \sigma \right) \leq K^2 \left( \frac{2^{i-l_1}\kappa}{\lambda^2} \sigma \right)^{2n-2+2\beta} \leq K^2 \sigma^{2n-2+2\beta}. \]

If (B) is true, then we proceed to Step 2.
Step 2. Since we are assuming $R(y_l) < \frac{2^{i-l} \kappa}{\lambda^2} \sigma$, there exists $l_2 \in \{k+1, ..., l_1-1\}$ so that

$$R(y_m) < \frac{2^{i-m} \kappa}{\lambda^2} \sigma, \forall m = l_2 + 1, ..., l_1.$$ 

Thus, we can apply Corollary 24 to obtain

$$|y_m - y_{m-1}| < \kappa R(y_m) < \frac{2^{i-m} \kappa^2}{\lambda^2} \sigma, \forall m = l_2 + 1, ..., l_1.$$ 

Hence

$$B_{y_m} \left( \frac{2^{i-l_1+1} \kappa^2}{\lambda^2} \sigma \right) \subset \cdots \subset B_{y_{m-1}} \left( \frac{2^{i-l_2} \kappa^2}{\lambda^2} \sigma \right).$$ 

Combined with (53), this implies

$$E_x(\sigma) \leq E_{y_m} \left( \frac{2^{i-l_2} \kappa^2}{\lambda^2} \sigma \right).$$

In the similar way as in Step 1, we prove $E_x(\sigma) \leq K^2 \sigma^{2n-2+2\beta}$ or we continue to Step 3 where we assume $R(y_{l_2}) < \frac{2^{i-l_2} \kappa^2}{\lambda^2} \sigma$. At Step $S$ of this procedure, we produce an integer $l_S \in \{k, ..., i\}$ and this procedure terminates finite number steps since $l_S > l_{S+1}$. Finally, observe that in order to prove our assertion, we must show that if $N$ is the number of step taken and if $y_{l_N} = y_k$ then the case corresponding to (B) (i.e. the case that $R(y_{l_N}) < \frac{2^{i-l_N} \kappa^2}{\lambda^2} \sigma$ ) does not occur. This is true because $R(y_{l_N}) = R(y_k) = \infty$ and this completes the proof. Q.E.D.

By modifying Morrey’s energy decay argument, we show that the energy growth estimate in Proposition 27 implies Hölder continuity.

**Proposition 28** Let $B$ be a dimension-$n$, codimension-$(n-k)$ local model, $g$ be an uniformly elliptic Lipschitz metric defined on $B(r)$ and $f : (B(r), g) \to Y$ a harmonic map. Suppose there exists $R > 0$ so that for every $x \in B(x)$,

$$E_x(\sigma) \leq K^2 \sigma^{2n-2+2\beta}, \forall x \in B(r/2), \sigma < R. \quad (54)$$

Then

$$d(f(x), f(y)) \leq C|x - y|^\beta, \forall x, y \in B(r/2)$$

with $C$ depending on $K$, $R$ and the geometry of $B$.

**Proof.** Without loss of generality, we will consider the case when $|x - y| < R$. We first restrict our attention to the case when $B(r)$ is a codimension-1 local model. Let $x, y \in B(r_0)$ with $|x - y| = \rho \leq r - r_0$. Let $\tilde{x}_t$, $0 \leq t \leq 1$, be the point which is fraction $t$ along the (Euclidean) geodesic from $x$ to $y$. Let $\Omega \subset B(r)$ so that $x, y \in \Omega$. First, we define the following: we say that $x$ and $y$ are geometrically related in $\Omega$ if $B_{\frac{2^t \sigma}{\lambda^2}}(\frac{2^t \sigma}{\lambda^2}) \cap \Omega$ is isometric to $B_{\frac{2^{i-t} \kappa^2}{\lambda^2}}(\frac{2^{i-t} \kappa^2}{\lambda^2}) \cap \Omega$ scaled
by a factor of $t$. Of course in Euclidean space, all points are geometrically related. But on a complex, this is not true if one passes through a singular point. Since $B(r)$ is a dimension-$n$, codimension-1 local model, there exist wedges $W_1$ and $W_2$ so that $x_t$ is contained in $W = W_1 \cup W_2$. Since $W$ is a disk, $x$ and $y$ are geometrically related in $W$; more specifically, if we set $\tilde{B}_t := B_2 \left( \frac{t \rho}{2} \right) \cap W$, then $\tilde{B}_t$ is isometric to $\tilde{B}_1$ scaled by a factor of $t$. For any $x' \in \tilde{B}_1$, let $x_t$, $0 \leq t \leq 1$, be an arclength parameterized geodesic from $x$ to $x'$. Since $|x - x'| \leq |x - x_0 + x_0 - x'| \leq \rho$, we have that the tangent vector $\dot{x}_t$ has length $\leq \rho$. Therefore,

$$d(f(x), f(x')) = \int_0^1 |f_\ast(\dot{x}_t)| \, dt \leq \rho \int_0^1 |\nabla f|(x_t) \, dt.$$ 

Let $C_2$ be sufficiently large so that the volume of a ball of radius $\tau$ in $B(r)$ is less than $C_2 \tau^n$. We integrate the above inequality over $x' \in \tilde{B}_1$. Note that we can switch the order of integration in the second inequality because $\tilde{B}_t$ is isometric to $\tilde{B}_1$ scaled by a factor of $t$.

$$\int_{\tilde{B}_1} d(f(x), f(x')) d\mu(x') \leq \rho \int_0^1 t^{\frac{1}{n}} \int_{B_2(\frac{t \rho}{2})} |\nabla f|(w) \, d\mu(w) \, dt \leq \rho \int_0^1 t^{\frac{1}{n}} \left( \int_{B_2(\frac{t \rho}{2})} |\nabla f|^2 \, d\mu \right)^{1/2} \, dt \leq \sqrt{C_1 C_2} \rho \int_0^1 t^{\frac{n}{2}} \left( \rho t + n - \frac{n + \beta}{2} \right) \, dt = \frac{\sqrt{C_1 C_2}}{\beta} \rho^{n+\beta}. $$

In the above, we are using the energy decay assumption (54). Similarly,

$$\int_{\tilde{B}_1} d(f(x'), f(y)) d\mu(x') = \frac{\sqrt{C_1 C_2}}{\beta} \rho^{n+\beta}. $$
By the triangle inequality and letting \( V = \text{volume}(\hat{B}_1) \),
\[
d(f(x), f(y)) = \frac{1}{V} \int_{\hat{B}_1} d(f(x), f(y)) d\mu(x')
\leq \frac{1}{V} \left( \int_{\hat{B}_1} d(f(x), f(x')) d\mu(x') + \int_{\hat{B}_1} d(f(x'), f(y)) d\mu(x') \right)
\leq C\rho^3
\]
with \( C \) dependent only on \( C_1, C_2 \) and \( \beta \). This proves the required Hölder continuity in the case \( B(r) \) is a dimension-\( n \), codimension-1 local model.

We proceed by induction and assume that the asserted Hölder continuity can be proven whenever \( B(r) \) is a dimension-\( n \), codimension-(\( k-1 \)) local model. Let \( B(r) \) be a codimension-\( k \) local model, \( x, y \in B(r_0) \) and \( x' \) and \( y' \) the closest point in \( D(r_0) \) to \( x \) and \( y \) respectively. If \( x = x' \) and \( y = y' \) or \( x' = y \), then \( x \) and \( y \) are geometrically related. Hence, the proof above for the case when \( B(r) \) is a codimension-1 local model can be used to show
\[
d(f(x), f(y)) \leq C|x - y|^3 \quad \text{whenever } x, y \in D(r_0) \text{ or } x' = y \in D(r_0). \tag{55}
\]
If \( y = y' \in D(r_0) \), then we can use (55) to show
\[
d(f(x), f(y)) \leq d(f(x), f(x')) + d(f(x'), f(y))
\leq C|x - y'|^3 + C|x' - y|^3
\leq 2C|x - y|^3
\]
and hence
\[
d(f(x), f(y)) \leq 2C|x - y|^3 \quad \text{whenever } y \in D(r_0). \tag{56}
\]
Now let \( x, y \) be arbitrary points of \( B(r_0) \) and assume without the loss of generality that \( |x - x'| \geq |y - y'| \). If \( |x - y| \leq |y - y'| \), then \( B_r(\hat{x}_{\frac{1}{2}}) \) does not intersect \( D(r_0) \) and the result follows directly from the inductive assumption, i.e. the Hölder continuity in the case when \( B(r) \) is a dimension-\( n \), codimension-(\( k-1 \)) local model. So assume \( |x - y| > |y - y'| \). Then
\[
|x - y'| \leq |x - y| + |y - y'| \leq 2|x - y|.
\]
Thus, by using (55) and (56), we see that
\[
d(f(x), f(y)) \leq d(f(x), f(y')) + d(f(y'), f(y))
\leq 2C|x - y'|^3 + |y' - y|^3
\leq 2C(|x - y| + |y - y'|)^3 + |y - y'|^3
\leq (2^{1+\beta}C + 1)|x - y|^3.
\]
Q.E.D.
Theorem 29 Let $B$ be a local model and $g$ be an uniformly elliptic Lipschitz metric defined on $B(r)$. Let $f : (B(r), g) \to Y$ be a harmonic map and $\alpha_x$ the order of $f$ at $x$. If $\alpha_x \geq \alpha$ for all $x \in B(r/2)$, then there exists $C$ depending only on the Lipschitz bound and ellipticity constant of $g$, $E_f$ and the geometry of $B$ so that
\[
d(f(x), f(y)) \leq C|x - y|^\alpha \quad \forall x, y \in B(r/2).
\]
By the geometry of $B(r)$, we mean the dimension of $B(r)$, the number of wedges of $B(r)$ as well as the wedge angles of $B(r)$.

**Proof.** Thus the result follows immediately from Lemma 22, Proposition 27 and Proposition 28. q.e.d.

Theorem 30 Let $B$ be a local model and $g$ be an uniformly elliptic Lipschitz metric defined on $B(r)$. If $f : (B(r), g) \to Y$ is a harmonic map, then there exist $C$ and $\gamma$ so that
\[
d(f(x), f(y)) \leq C|x - y|^\gamma \quad \forall x, y \in B(r/2).
\]
Here, $C$ and $\gamma$ only depend on the Lipschitz bound and the ellipticity constant of $g$, $E_f$ and the geometry of $B$. By the geometry of $B(r)$, we mean the dimension of $B(r)$, the number of wedges of $B(r)$ as well as the wedge angles of $B(r)$.

**Proof.** As in the proof of Theorem 29, this follows immediately from Lemma 22, Proposition 27 and Proposition 28. q.e.d.

5 Convergence in the pull back sense

Given a map $u : B(r) \to (Y, d)$, we recall the following construction of [KS2]. First, we let $\Omega_0 = B(r)$, $u_0 = u$ and $d_0 : \Omega \times \Omega \to R^+ \cup \{0\}$ be the pseudodistance function $d_0(x, y) = d(u_0(x), u_0(y))$. Next, we inductively define $\Omega_{i+1} = \Omega_i \times \Omega_i \times [0, 1]$ and identify $\Omega_i$ as a subset of $\Omega_{i+1}$ by the inclusion map $x \mapsto (x, x, 0)$. Extend $u_i : \Omega \to (Y, d)$ to $u_{i+1} : \Omega \to (Y, d)$ by
\[
u_{i+1}(x, y, \lambda) = (1 - \lambda)u_i(x) + \lambda u_i(y)
\]
and let
\[
d_{i+1}(x, y) = d(u_{i+1}(x), u_{i+1}(y)).
\]
Then
\[
d_{i+1}((x, x, 0), (y, y, 0)) = d_i(x, y)
\]
\[
d_{i+1}((x, y, \lambda), (x, y, \mu)) = |\lambda - \mu|d_i(x, y)
\]
Let $\Omega^\infty = \cup \Omega_i$ and define $u^\infty : \Omega^\infty \to (Y, d)$ by setting $u^\infty = u_i$ on $\Omega_i$. With $d^\infty(x, y) := d(u^\infty(x), u^\infty(y))$, define $(Y^\infty, d^\infty)$ as the completion of the quotient space from $(\Omega^\infty, d^\infty)$ and let $\pi : \Omega^\infty \to Y^\infty$ be the natural projection map. Equation (57) implies that the metric space $(Y^\infty, d^\infty)$ is an NPC space. The unique extension of $u^\infty$ to $Y^\infty$ is an isometry $U : (Y^\infty, d^\infty) \to C(u(B(r))) \subset Y$ to the closed convex hull of the image of $u$. Furthermore, if $\iota : B(r) = \Omega_0 \to \Omega^\infty$ is the inclusion map, then $u = U \circ \iota$. (cf. [KS2])

**Definition 31** Let $v_k : B(r) \to (Y_k, d_k)$ be a sequence of maps to NPC spaces. We say $v_k$ converge to $v_*$ in the pullback sense if there exists a pseudodistance function $d_* : \Omega^\infty \times \Omega^\infty \to \mathbb{R}^+ \cup \{0\}$ with the following property. Let $(Y_*, d_*)$ be the completed quotient space from $(\Omega^\infty, d^\infty)$ and $\pi : \Omega^\infty \to Y^\infty$ the natural projection map. Furthermore, let $v_k = u$ in the above paragraph and let $d_k^\infty : \Omega^\infty \times \Omega^\infty \to \mathbb{R}^+ \cup \{0\}$ the corresponding pullback distance function of $v_k^\infty$ (which equals $u^\infty$ above). Then $d_k^\infty$ converges pointwise to $d_*$ and $v_* = \pi \circ \iota$.

**Remark.** If we let $v_* = u$ with $u$ as in the paragraph preceding the definition above and $d_*, d^\infty$ resp.) the corresponding pullback distance function of $v_*, d^\infty$ resp.), then $d_* = d^\infty$.

**Definition 32** Suppose $v_k$ converge to $v_*$ in the pullback sense. Let $d_{k,i}$ $d^\infty$ resp.) be the corresponding pullback distance function to $v_k$ $\Omega_i \to (Y_k, d_k)$ $v_{*,i} : \Omega_i \to (Y_*, d_*)$ resp.). We say that the convergence is locally uniform if the convergence of $d_{k,i}$ to the limit $d_{*,i}$ is uniform on each compact subset of $\Omega_i \times \Omega_i$. In this case, we also say that $v_*$ is a locally uniform limit of $v_k$.

**Proposition 33** Let $v_k : B(r) \to (Y_k, d_k)$ be a sequence of maps to NPC spaces for which there is uniform modulus of continuity control, i.e. assume for each $x \in B(r)$ and $R > 0$ there is a positive function $\omega(x, R)$ which is monotone in $R$, satisfying

$$\lim_{R \to 0} \omega(x, R) = 0,$$

and so that for each $k \in \mathbb{Z}$

$$\max_{y \in B(x, R)} d(v_k(x), v_k(y)) \leq \omega(x, R).$$

Then there is a NPC space $(Y_*, d_*)$ and a subsequence $v_{k'}$ of the $v_k$ which converges locally uniformly in the pullback sense to a limit map $v_* : B(r) \to (Y_*, d_*)$, and $v_*$ satisfies the same modulus of continuity estimates. Here, $(Y_*, d_*)$ is the completed quotient of $(\Omega^\infty, d^\infty)$ where $d^\infty = \lim_{k' \to \infty} d_{k',\infty}$.
Proof. The proposition follows from the argument of the proof of Lemma 3.1 and Proposition 3.7 in [KS2] since the fact that $B(r)$ is not a Riemannian domain plays no consequence in the argument. Q.E.D.

6 The tangent map

Let $B$ be a dimension-$n$, codimension-$\nu$ local model and $g$ be a normalized Lipschitz metric on $B(1)$ with ellipticity constant $\lambda$. Given $f : B(r) \to (Y, d)$ and $\lambda$ sufficiently small so that $\lambda r < 2$, define the $\lambda$-blow up map $f_\lambda : B^2 \to (Y, d_\lambda)$ by setting:

$$
g_\lambda(x) = g(\lambda x)$$

$$
\mu_\lambda = (\lambda^{1-n}I(\lambda))^{1/2}
$$

$$
d_\lambda(p, q) = \mu_\lambda^{-1}d(p, q)
$$

$$
f_\lambda(x) = f(\lambda x).
$$

Definition 34 If there exists $\lambda_k \to 0$ and a NPC space $(Y^*, d_*)$ so that $f_{\lambda_k}$ converges locally uniformly in the pullback sense to $f_* : B(1) \to (Y^*, d_*)$, then $f_*$ is called a tangent map of $f$.

Theorem 35 A harmonic map $f : (B, g) \to (Y, d)$ has a non-constant tangent map $f_*$ which satisfies

$$
d(f_*(x), f_*(y)) \leq C'|x - y|^{\gamma}
$$

where $C'$ and $\gamma$ are only dependent on the Lipschitz bound of $g$ and $E^f$.

Proof. By change of variables

$$
\int_{B(\sigma)} |\nabla f_{\lambda, g_\lambda}|^2 d\mu_{g_\lambda} = \mu_\lambda^{-2}\lambda^{2-n} \int_{B(\lambda \sigma)} |\nabla f|^2 d\mu_g
$$

and

$$
\int_{\partial B(\sigma)} d^2(f, f_*(0)) d\Sigma_{g_\lambda} = \mu_\lambda^{-2}\lambda^{1-n} \int_{\partial B(\lambda \sigma)} d^2(f, f(0)) d\Sigma_{g}.
$$

Thus, the definition of $\mu_\lambda$ implies

$$
\int_{\partial B(1)} d^2(f_*, f_*(0)) d\Sigma_{g_\lambda} = 1
$$

and Corollary 19 implies

$$
\lim_{\lambda \to 0} \frac{\int_{B(1)} |\nabla f_{\lambda, g_\lambda}|^2 d\mu_{g_\lambda}}{\int_{\partial B(1)} d^2(f_*, f_*(0)) d\Sigma_{g_\lambda}} = \lim_{\lambda \to 0} \frac{\lambda E(\lambda)}{I(\lambda)} = \alpha.
$$

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Consequently, by choosing $\lambda$ sufficiently small, we have
\[
\int_{B(1)} |\nabla f_\lambda|^2 d\mu_{g_\lambda} \leq 2\alpha.
\] (58)

Since $g_{\alpha\beta}$ is Lipschitz, we have
\[
|g_{\alpha\beta}(x) - \delta_{\alpha\beta}| \leq c|x| \quad \text{and} \quad |g^{\alpha\beta}(x) - \delta_{\alpha\beta}| \leq c|x|.
\] (59)

Hence,
\[
|(g_\lambda)_{ij}(x) - \delta_{\alpha\beta}| \leq c\lambda|x| \quad \text{and} \quad |(g_\lambda)^{ij} - \delta_{\alpha\beta}| \leq c\lambda|x|.
\] (60)

Therefore, there exists an uniform Lipschitz bound of the family of metrics $\{g_\lambda\}$ independent of $\lambda$. This implies
\[
d_\lambda(f_\lambda(x), f_\lambda(y)) \leq C|x - y|^{\gamma} \quad \forall x, y \in B(r),
\]
where $C$ and $\gamma$ are independent of $\lambda$ by Theorem 30. Thus, we have uniform modulus of continuity control for the sequence $f_\lambda$, then by Proposition 33, there exists a sequence $\lambda_k \to 0$ and a NPC space $(Y, d_*)$ so that $f_{\lambda_k}$ converges locally uniformly in the pullback sense to a limit map $f_* : B(1) \to (Y, d_*)$. The fact that $f_*$ is non-constant follows immediately from the proof of Proposition 3.3 of [GS]. Q.E.D.

**Lemma 36** Let $f : (B(r), g) \to (Y, d)$ be a harmonic map and let $f_\lambda : B(1) \to (Y, d_\lambda)$ be the $\lambda$-blow up map. Let $h_\lambda : B(1) \to (Y, d_\lambda)$ be a map which is harmonic with respect to the Euclidean domain metric and with the boundary condition $h_\lambda|_{\partial B(1)} = f_\lambda|_{\partial B(1)}$. Then
\[
(1 - c\lambda) \delta_\lambda E^{f_\lambda} \leq g_\lambda E^{f_\lambda} \leq (1 + c\lambda) \delta_\lambda E^{f_\lambda}
\] (61)

and
\[
(1 - c\lambda) \delta_\lambda E^{h_\lambda} \leq g_\lambda E^{h_\lambda} \leq (1 + c\lambda) \delta_\lambda E^{h_\lambda}.
\] (62)

**Proof.** By inequality (60), we have for any tangent vector $v$,
\[
|g_\lambda(v, v) - \delta(v, v)| \leq c\lambda \delta(v, v)
\]

and hence
\[
(1 - c\lambda) \sum_{i,j=1}^{n} \delta^{\alpha\beta} \frac{\partial f_\lambda}{\partial x_i} \cdot \frac{\partial f_\lambda}{\partial x_j} \leq \sum_{i,j=1}^{n} (g_\lambda)^{ij} \frac{\partial f_\lambda}{\partial x_i} \cdot \frac{\partial f_\lambda}{\partial x_j} \leq (1 + c\lambda) \sum_{i,j=1}^{n} \delta^{\alpha\beta} \frac{\partial f_\lambda}{\partial x_i} \cdot \frac{\partial f_\lambda}{\partial x_j}
\] (63)

and
\[
(1 - c\lambda) \sum_{i,j=1}^{n} \delta^{\alpha\beta} \frac{\partial h_\lambda}{\partial x_i} \cdot \frac{\partial h_\lambda}{\partial x_j} \leq \sum_{i,j=1}^{n} (g_\lambda)^{ij} \frac{\partial h_\lambda}{\partial x_i} \cdot \frac{\partial h_\lambda}{\partial x_j} \leq (1 + c\lambda) \sum_{i,j=1}^{n} \delta^{\alpha\beta} \frac{\partial h_\lambda}{\partial x_i} \cdot \frac{\partial h_\lambda}{\partial x_j}
\] (64)
The assertion follows immediately. Q.E.D.

In particular, Lemma 36 implies that

$$\delta E^{h_\lambda} \leq \frac{\delta E^{f_\lambda}}{1 - c\lambda} \leq g_\lambda E^{f_\lambda} \leq \frac{2\alpha}{1 - c\lambda}. \quad (65)$$

Thus, Proposition 33 and Theorem 30 imply that there exists a subsequence of $\lambda_k$ (which we will still denote $\lambda_k$ by an abuse of notation) and a NPC space $(Y_\ast, d_\ast)$ so that $h_{\lambda_k}$ converge locally uniformly in the pullback sense to $h_\ast : B(1) \to (Y_\ast, d_\ast)$. Set $h_k := h_{\lambda_k}$, $f_k := f_{\lambda_k}$ and $g_k = g_{\lambda_k}$. Furthermore, let $d_k(x, y) = d_{\lambda_k}(f_k(x), h_k(x))$ and $d_k(x, y) = d_{\lambda_k}(h_k(x), h_k(y))$. Then in any compactly contained subset of $B(1) \times B(1)$, $d_k, d_k'$ converge uniformly to (the restriction to $B(1) = \Omega_0$ of) $d_\ast, d_\ast'$ respectively.

**Proposition 37** The pseudodistance functions $d_\ast$ and $d_\ast'$ above are equal. Consequently, $f_\ast = h_\ast$ and $f_\ast$ is a locally uniform limit of a sequence $h_k$ of harmonic maps with respect to the Euclidean metric.

**Proof.** By the repeated use of the triangle inequality,

$$|d_k(x, y) - d_k(x, y)| \leq d_{\lambda_k}(f_k(x), h_k(x)) + d_{\lambda_k}(f_k(y), h_k(y)).$$

Therefore, for any $r < 1$, the Lebesgue dominated convergence theorem and the Poincaré inequality (cf. Theorem 10) implies

$$\int_{B(r)} \int_{B(r)} |d_\ast(x, y) - d_\ast(x, y)|^2 dx = \lim_{k \to 0} \int_{B(r)} \int_{B(r)} |d_k(x, y) - d_k(x, y)|^2 dx$$

$$= 4\text{vol}(B(r)) \lim_{k \to 0} \int_{B(r)} d_{\lambda_k}^2(f_k(x), h_k(x)) dx$$

$$\leq 4C \lim_{k \to 0} \int_{B(r)} |\nabla d_{\lambda_k}^2(f_k(x), h_k(x))| dx. \quad (66)$$

Equations (61) and (62) imply

$$\delta E^{f_\lambda} \leq \frac{1}{1 - c\lambda} g_\lambda E^{f_\lambda} \leq \frac{1 + c\lambda}{1 - c\lambda} \delta E^{h_\lambda}. \quad (65)$$

Therefore, if we let $w = \frac{1}{2}f_\lambda + \frac{1}{2}h_\lambda$,

$$2 \delta E^w \leq \delta E^{f_\lambda} + \delta E^{h_\lambda} - \frac{1}{2} \int_{B(r)} |\nabla d_{\lambda_k}^2(f_\lambda, h_\lambda)| d\mu$$

$$= 2 \delta E^{h_\lambda} + O(\lambda) - \frac{1}{2} \int_{B(r)} |\nabla d_{\lambda_k}^2(f_\lambda, h_\lambda)| d\mu$$

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by equation (2.2iv) of [KS1]. Since $E^{h,\lambda} \leq E^{w}$, this in turn implies
\[
\int_{B(r)} |\nabla d_{\lambda}^{2}(f_{\lambda}, h_{\lambda})| d\mu \to 0
\]
as $\lambda \to 0$. This, combined with equation (66) and continuity of $d_{\ast}$ and $\bar{d}_{\ast}$, shows that $d_{\ast}(x, y) = \bar{d}_{\ast}(x, y)$ which in turn implies that $(Y_{\ast}, d_{\ast}) = (\bar{Y}_{\ast}, \bar{d}_{\ast})$ and $h_{\ast} = f_{\ast}$. Q.E.D.

**Lemma 38** Assuming that the directional energies of $h_{k}$ converge to those of $f_{\ast}$, the tangent map $f_{\ast} : B(1) \to Y_{\ast}$ is homogenous of order $\alpha$ where $\alpha$ is the order of $f$ at $0$, i.e.
\[
d_{\ast}(f_{\ast}(x), f_{\ast}(0)) = |x|^\alpha d_{\ast}\left(\frac{x}{|x|}, f_{\ast}(0)\right)
\]
and the image of $t \mapsto f_{\ast}(tx)$, $0 \leq t \leq 1$, is a geodesic.

**Proof.** Using (5) and (10) with $f$ replaced by $h_{k}$, noting that the remainder in (5) is 0 because the domain is Euclidean, and using the convergence of $h_{k}$ and its directional energies to $f_{\ast}$ and its directional energies, we have
\[
(E^{f_{\ast}}(\sigma))' = 2 \int_{\partial B(\sigma)} \left| \frac{\partial f_{\ast}}{\partial r} \right|^2 d\Sigma
\]
and
\[
2E^{f_{\ast}}(\sigma) \leq \int_{\partial B(\sigma)} d(f_{\ast}, f_{\ast}(0)) \frac{\partial}{\partial r} d(f_{\ast}, f_{\ast}(0)) d\Sigma.
\]
We next claim
\[
\left( \log \left( \frac{\sigma E^{f_{\ast}}(\sigma)}{I f_{\ast}(\sigma)} \right) \right)'
\geq \frac{2}{E^{f_{\ast}}(\sigma) I f_{\ast}(\sigma)} \left[ \left( \int_{B(\sigma)} d^2(f_{\ast}, f_{\ast}(0)) d\Sigma \right) \left( \int_{B(\sigma)} \left| \frac{\partial f_{\ast}}{\partial r} \right|^2 d\Sigma \right) \right.
\]
\[
- \left. \int_{\partial B(\sigma)} d(f_{\ast}, f_{\ast}(0)) \frac{\partial}{\partial r} d(f_{\ast}, f_{\ast}(0)) d\Sigma \right] \geq 0. \quad (67)
\]
This follows from (8), (13) and (18) applied to the harmonic map $h_{k}$ (without the error term due to the fact that $h_{k}$ is harmonic for the Euclidean domain metric) and the assumption that the directional energies of $h_{k}$ converge to those of $f_{\ast}$. On the other hand, our assumption on the convergence of directional energies
implies that
\[
\frac{\sigma E^{f^*}(\sigma)}{H^*(\sigma)} = \lim_{k \to \infty} \frac{\sigma E^{f^*}_k(\sigma)}{I^{f^*_k}(\sigma)}
\]
\[
= \lim_{k \to \infty} \frac{\sigma \mu^2 \lambda^{2-n} \sigma E^{f_k}(\lambda_k \sigma)}{\mu \lambda^{1-n} \sigma E^{f_k}(\lambda_k \sigma)}
\]
\[
= \lim_{k \to \infty} \frac{\sigma \lambda^2 \sigma E^{f_k}(\lambda_k \sigma)}{\sigma E^{f_k}(\lambda_k \sigma)}
\]
\[
= \alpha.
\]
Thus,
\[
0 = \left( \log \frac{\sigma E^{f^*}(\sigma)}{H^*(\sigma)} \right)'
\]
\[
= \frac{2}{E^{f^*}(\sigma)H^*(\sigma)} \left[ \left( \int_{B(\sigma)} d^2(f^*, f^*_0) d\Sigma \right) \left( \int_{\partial B(\sigma)} \left| \frac{\partial f_*}{\partial r} \right|^2 d\Sigma \right) \right.
\]
\[
- \int_{\partial B(\sigma)} \left( d(f^*, f^*_0) \frac{\partial}{\partial r} d(f^*, f^*_0) d\Sigma \right)^2 \right].
\]
Hence,
\[
\frac{\partial}{\partial r} d(f^*, f^*_0) = \left| \frac{\partial f_*}{\partial r} \right| \text{ a.e.}
\]
and
\[
2 \int_{B(\sigma)} |\nabla f_*|^2 d\mu = \int_{\partial B(\sigma)} \frac{\partial}{\partial r} \left( d^2(f^*, f^*_0) \right) d\Sigma.
\]
We can now follow the proof of Proposition 3.1 [GS] to show the homogeneity of $f_*$. Q.E.D.

**Lemma 39** Let $f_* : B(1) \to (Y_*, d_*)$ be a homogeneous map of order $\alpha$. (See definition of homogeneity in Lemma 38). Then there exists a metric space $(C, \hat{d})$ and map $\hat{f}_* : B(1) \to C$ so that energy density of $f_*$ is equal to that of $\hat{f}_*$ and for every $x, y \in \partial B(1)$ and $x' = tx, y' = ty$, we have

\[
\hat{d}(\hat{f}_*(x'), \hat{f}_*(y')) = t^\alpha \hat{d}(\hat{f}_*(x), \hat{f}_*(y)). \tag{68}
\]

**Proof.** Let $C$ be the disjoint union of geodesics from $f(0)$ to $f(x)$ for each $x \in \partial B(1)$ with $f(0)$ identified. We define a distance function $\hat{d}$ on $C$ in the following way. Let $P, Q \in C$ and suppose that $P$ (resp. $Q$) is the point on the geodesic $\gamma$ (resp. $\sigma$) from $f(0)$ to $f(x)$ (resp. $f(y)$) at a distance $r$ (resp. $s$) from $f(0)$, where $x, y \in \partial B(1)$. We first define the angle $\theta$ between $\gamma$ and $\sigma$ by

\[
\cos \theta = \frac{d^2_*(f_*(0), f_*(x)) + d^2_*(f_*(0), f_*(y)) - d^2_*(f_*(x), f_*(y))}{2 d_*(f_*(0), f_*(x))d_*(f_*(0), f_*(y))}
\]
and set
\[ \hat{d}^2(P, Q) = r^2 + s^2 - 2rs \cos \theta, \quad \hat{f}(x) = f(x). \]

By definition of \( \hat{d} \), we see that (68) holds. Therefore,
\[ \hat{d}(\hat{f}_* (x), \hat{f}_* (y)) = d_*(f_*(x), f_*(y)) \]
whenever \( x, y \) lie on the same geodesic from \( f(0) \) or whenever \( x, y \in \partial B(1). \)

Therefore, for any \( x \in \partial B(r) \) and any vector \( V \) normal to \( \partial B(r) \),
\[ |f_*(V)|^2(x) = |\hat{f}_*(V)|^2(x). \]

Furthermore, the same holds for any \( x \in \partial B(1) \) and \( V \) tangential to \( \partial B(1). \) For a.e \( x = (r, \theta) \in B(r) \) and a vector \( V \) tangential to \( \partial B(r) \), [KS1] Lemma 1.9.4 says that
\[ |\hat{f}_*(V)|^2(r, \theta) = \lim_{\epsilon \to 0} \frac{\hat{d}_2(\hat{f}_*(r, \theta), \hat{f}_*(r, \theta + \epsilon V))}{\epsilon^2} \]
where \( (r, \theta) \) is the polar coordinates of \( x \) and \( \epsilon \mapsto \theta + \epsilon V \) is the flow along \( \partial B(r) \) defined by \( V \). Same equalities are true when \( V \) is normal to \( \partial B(r) \) by a similar argument. Q.E.D.

7 Harmonic maps from a flat domain

Let \( B \) have wedges \( W_k \subset \mathbb{R}^n \) \((k = 1, \ldots, l)\). Recall that coordinates \((x^1, \ldots, x^n)\) of \( \mathbb{R}^n \) are arranged so that \( D \) is given by \( x^{n-\nu+1} = \ldots = x^n = 0 \). In this section, we show that harmonic maps \( h : (B(r), \delta) \to (Y, d) \) are Lipschitz in the direction parallel to \( D(r) = D \cap B(r) \).

**Lemma 40** Let \( h : (B(1), \delta) \to (Y, d) \) be a harmonic map. Let \( V \) be a unit vector parallel to \( D(r) \) and let \( H(x) = h(x + \epsilon V) \) for \( 0 < \epsilon << 1 \). Then
\[ 0 \leq -\int_{B(1)} \nabla \eta \cdot \nabla d^2(h, H) d\mu \]
for \( \eta \in C_c^\infty(B(1 - \epsilon)) \).
Proof. Define a map $h_\eta : B(1 - \epsilon) \to \mathbb{R}$ by setting

$$h_\eta(x) = (1 - \eta(x))h(x) + \eta(x)H(x).$$

Here, $(1 - t)P + tQ$ for $P, Q \in Y$ denotes the point on the unique geodesic between $P$ and $Q$ at a distance $td(P, Q)$ from $P$ and $(1 - t)d(P, Q)$ from $Q$.

Since $\text{spt}(\eta) \subset B(1 - \epsilon)$, we see that $h_\eta|_{\partial B(1 - \epsilon)} = h|_{\partial B(1 - \epsilon)}$ and $h_1 - \eta|_{\partial B(1 - \epsilon)} = H|_{\partial B(1 - \epsilon)}$.

By following the proofs of Lemma 2.4.1 and 2.4.2 of [KS1], we see that $h_\eta, h_1 - \eta \in W^{1,2}(B(r))$, and on each wedge $W_k, j = 1, ..., N$, we have

$$\int_{B(r) \cap W_k} |\nabla h_\eta|^2 + \int_{B(r) \cap W_k} |\nabla h_1 - \eta|^2 \leq \int_{B(r) \cap W_k} |\nabla h|^2 + \int_{B(r) \cap W_k} |\nabla H|^2 - 2\int_{B(r) \cap W_k} \eta \cdot \nabla d^2(h, H) + \int_{B(r) \cap W_k} Q(\eta, \nabla \eta)$$

where $Q(\eta, \nabla \eta)$ consists of integrable terms which are quadratic in $\eta$ and $\nabla \eta$.

Taking the sum over $k = 1, ..., l$ and noting that

$$\int_{B(1 - \epsilon)} |\nabla h|^2 \leq \int_{B(1 - \epsilon)} |\nabla h_\eta|^2$$

and

$$\int_{B(1 - \epsilon)} |\nabla H|^2 \leq \int_{B(1 - \epsilon)} |\nabla h_1 - \eta|^2,$$

by the harmonicity of $h$, we deduce

$$0 \leq -2\int_{B(1 - \epsilon)} \eta \cdot \nabla d^2(h, H) + \int_{B(1 - \epsilon)} Q(\eta, \nabla \eta).$$

By replacing $\eta$ by $t\eta$, dividing by $t$ and letting $t \to 0$, we obtain (69). Q.E.D.

For a point $x$ in $B(1)$ belonging to a wedge $W$, recall that $R(x)$ is defined in the following way: if $x$ is in the interior of $W$, $R(x)$ is the distance of $x$ to $\partial W \cup \partial B(1)$. If $x$ is a point on a face $F$ of $W$, we let $R(x)$ be the smaller of the distance of $x$ to the closest face that is not $F$ or the distance of $x$ to $\partial B(1)$.

**Lemma 41** Let $h : (B(1), \delta) \to Y$ be a harmonic map and $V$ be a unit vector parallel to $D(1)$. For any $r \in (0, 1)$ and $x \in B(r)$, there exists constant $C$ depending on $E^1, r$ and the geometry of $B$ so that

$$|h_*(V)|^2(x) \leq CE^h.$$
Proof. Let $H$ be as in the proof of Lemma 40. For $x \in B(1)$ and $\sigma < R(x)$, let $\eta$ approximate the characteristic function of $B_\sigma(\sigma)$ in (69) to obtain
\[
\int_{\partial B_\sigma(\sigma)} \frac{\partial}{\partial r} d^2(h, H) d\Sigma \geq 0.
\]
Let
\[
J(\sigma) = \int_{\partial B_\sigma(\sigma)} d^2(h, H) d\Sigma \quad \text{and} \quad K(\sigma) = \int_{B_\sigma(\sigma)} d^2(h, H) d\mu
\]
for $0 < \sigma < R(x)$. Then
\[
J'(\sigma) = \int_{\partial B_\sigma(\sigma)} \frac{\partial}{\partial r} d^2(h, H) d\Sigma + \frac{n-1}{\sigma} J(\sigma) \geq \frac{n-1}{\sigma} J(\sigma).
\]
This implies that
\[
\left( \frac{J(\sigma)}{\sigma^{n-1}} \right)' \geq 0,
\]
and hence
\[
J(\tau) \leq J(\sigma) \sigma^{n-1}, \quad 0 < \tau \leq \sigma \leq R(x).
\]
Now integrate the above inequality from $\tau = 0$ to $\tau = \sigma$ to obtain
\[
K(\sigma) \leq \frac{\sigma J(\sigma)}{n} = \frac{\sigma K'(\sigma)}{n}.
\]
Thus,
\[
\left( \frac{K(\sigma)}{\sigma^n} \right)' = \frac{1}{\sigma^n} \left( K'(\sigma) - \frac{nK(\sigma)}{\sigma} \right) \geq 0.
\]
This implies $\sigma \mapsto \frac{K(\sigma)}{\sigma^n}$ is non-decreasing for $\sigma \leq R(x)$ and hence
\[
\frac{K(\sigma)}{\sigma^n} \leq \frac{K(R)}{R^n}, \quad 0 < \sigma \leq R \leq R(x).
\]
We then obtain
\[
d^2(h(x), H(x)) \leq \frac{CK(R)}{R^n}, \quad \text{for } \sigma \leq R.
\]
Divide by $\epsilon^2$ and let $\epsilon \to 0$ to get
\[
|h_*(V)|^2(x) \leq \frac{C}{R^n} \int_{B_\epsilon(R)} |h_*(V)|^2 d\mu \leq C'E^h.
\]
Q.E.D.
Lemma 42 Let $h : (B(1), \delta) \to Y$ be a harmonic map and $r \in (0,1)$. If $x, y$ are a pair of points in a wedge of $B(r)$ equidistant to $D(r)$, then
\[ d(h(x), h(y)) \leq L|x - y| \]
for some constant $L$ depending only on $E^h$, $r$ and the geometry of $B$.

Proof. Let $\gamma : [0,1] \to Y$ be a constant speed parameterization of the line between $x$ and $y$. Then by Lemma 41,
\[ d(h(x), h(y)) \leq \int_0^1 |h_*(\gamma'(t))| \, dt \leq \sqrt{C'E^h}|x - y|. \]
Q.E.D.

8 Lipschitz regularity

8.1 At a regular point

In this subsection, we use the results of Section 7 to give a new proof of the Lipschitz regularity of Korevaar-Schoen [KS1] and generalize their result for Lipschitz domain metrics. Recall that a dimension-$n$, codimension-0 local model is $B = \mathbb{R}^n$.

Lemma 43 Let $B$ be a dimension-$n$, codimension-0 local model, $g$ a normalized Lipschitz metric defined on $B(r)$, $(Y, d)$ a NPC space and $f : (B(r), g) \to (Y, d)$ a harmonic map. Then the order $\alpha$ of $f$ at 0 is $\geq 1$.

Proof. By Proposition 37, a tangent map $f_*$ of $f$ is a locally uniform limit of a sequence of harmonic maps $h_k$ from a Euclidean unit ball $B(1)$. The regularity result of [GS] implies that $h_k$ is locally Lipschitz with the local Lipschitz bound dependent on $E^{h_k}$ and the distance to $\partial B(1)$. Hence, so is $f_*$. By [KS2] Theorem 3.11, the energy densities of $h_k$ converge to those of $f_*$. By Lemma 38, $f_*$ is a homogeneous map of order $\alpha$. The homogeneity and the Lipschitz continuity of $f_*$ implies $\alpha \geq 1$. Q.E.D.

Theorem 44 Let $B$ be a dimension-$n$, codimension-0 local model, $g$ be a uniformly elliptic Lipschitz metric on $B(r)$, $(Y, d)$ a NPC space and $f : (B(r), g) \to (Y, d)$ be a harmonic map. Then $f$ is Lipschitz continuous in $B(\frac{r}{2})$ with the Lipschitz constant dependent on $E^f$, and the Lipschitz bound and the ellipticity constants of $g$.

Proof. For each $x \in B(1)$, the normalized map $f \circ L_x$ (cf. Proposition 4) has order $\geq 1$ at 0 by Lemma 43. Thus, the order of $f$ at $x$ is $\geq 1$. The result now follows from Theorem 29. Q.E.D.
8.2 At a codimension-1 singular point

A dimension-$n$, codimension-1 local model has wedges which are half spaces $x^n \geq 0$ and $D$ is a hyperplane given by the equation $x^n = 0$. We first prove some properties of harmonic maps from this local model equipped with the Euclidean metric $\delta$.

**Lemma 45** Let $h : (B(1), \delta) \to Y$ be a harmonic map. For every $\beta, r \in (0,1)$, there exists $B$ only dependent on $\beta, r$ and the total energy of the map $h$ so that

$$d(h(x), h(y)) \leq B|x - y|^{\beta}$$

for every $x, y \in B(r)$.

**Proof.** By Lemma 42, $h$ is Lipschitz when restricted to $D(t_0), t_0 = \frac{r+1}{2}$. Thus Hölder regularity of $h$ restricted to a wedge $W$ with any Hölder exponent $\beta \in (0,1)$ follows from the boundary regularity result of Serbinowski [Se] where the Hölder constant $B$ is only dependent on the choice of $\beta, r$ and the total energy of the map $h$. Q.E.D.

The next lemma gives an estimate of the energy decay of harmonic maps along an $\epsilon$-neighborhood.

**Lemma 46** Let $h_k$ be defined as in Section 6 (see the paragraph preceding Proposition 37) and fix $R \in (0,1)$. Set $D_\epsilon(r)$ to be the $\epsilon$-neighborhood of $D(r)$ in $B(r)$, i.e. it is the union of the $\epsilon$-neighborhoods of $D(r)$ in the wedges $W$ of $B(r)$ or equivalently

$$D_\epsilon(r) = \bigcup\{x = (x_1, \ldots, x_n) \in W : x_n \leq \epsilon\} \cap B(r)$$

where the union is over the wedges $W$. Then any $r \in (0, R)$, there exists constants $C, \delta > 0$, $k_0$ sufficiently large and $\epsilon_0 > 0$ sufficiently small (depending only on $R$) so that

$$E_{h_k}[D_{\epsilon}(r)] \leq C\epsilon^\delta, \forall k > k_0, \epsilon < \epsilon_0$$

**Proof.** Let $B_x(r)$ be a ball of radius $r$ centered at $x$. We will use the notation,

$$E^h_{h_k}(r) = \int_{B_x(r)} |\nabla h|^2 d\mu$$

and

$$I^h_{h_k}(r) = \int_{\partial B_x(r)} d^2(h, h(x))d\Sigma$$

for any map $h : B(r) \to (Y,d)$. Let $r_0 := 1 - R$. By Theorem 35, $f_*$ is a non-constant, continuous map and hence there exists $c_1 > 0$ so that

$$I^f_{x}(r_0) \geq 2c_1, \forall x \in D(R).$$
Thus, by the local uniform convergence, there exists \( k_0 \) so that
\[
I^{h_k}(r_0) \geq c_1, \forall x \in D(R), k > k_0.
\]
By (65), we may assume we have chosen \( k \) sufficiently large so that \( \lambda_{k_0} \in (0, \frac{1}{Nc}) \) and hence
\[
E^{h_k}(r_0) \leq E^{f_k}(1) \leq \frac{2\alpha}{1 - c\lambda_{k_0}}, \forall x \in D(R), k > k_0.
\]
Thus,
\[
\frac{r_0E^{h_k}(r_0)}{I^{h_k}(r_0)} \leq \frac{2r_0\alpha}{(1 - c\lambda_{k_0})c_1} =: c_2, \forall x \in D(R), k > k_0.
\]
By Corollary 21,
\[
\epsilon E^{h_k}(\epsilon) \leq c_3, \forall x \in D(R), k > k_0, \epsilon \leq r_0
\]
with \( c_3 \) depending on \( c_2 \). By Lemma 45,
\[
E^{h_k}(\epsilon) \leq \frac{c_3I^{h_k}(\epsilon)}{\epsilon} \leq \frac{c_3B^2\epsilon^{2\beta + n - 1}}{\epsilon} = c_4B^2\epsilon^{2\beta + n - 2}, \forall x \in D(R), k > k_0.
\]
Here, we have chosen \( \beta \in (1/2, 1) \). Since \( D_\epsilon(r) \) can be covered by \( \left( \frac{4r}{\epsilon} \right)^{n-1} \) number of \( (3\epsilon) \)-balls centered at points in \( D(r) \),
\[
E^{h_k}[D_\epsilon(r)] \leq c_4B^2(3\epsilon)^{2\beta + n - 2} \left( \frac{4r}{\epsilon} \right)^{n-1} = c_4B^2(4r)^{n-1}3^{2\beta - n - 2} \epsilon^{2\beta - 1}.
\]
The result follows from the fact that the choice of \( \beta \) implies \( 2\beta - 1 > 0 \). Q.E.D.

Lemma 47 For \( r \in (0, 1) \),
\[
\lim_{k \to 0} E^{h_k}(r) = E^{\ast}(r) \quad (70)
\]
and
\[
\lim_{k \to \infty} g_k E^{h_k}(r) = E^{\ast}(r), \quad (71)
\]
and the directional energies of \( f_k \) and \( h_k \) converge to that of \( f_\ast \).

Proof. By the regularity result of harmonic maps from smooth domains ([KS1] Theorem 2.4.6) or Theorem 44, \( h_k \) is uniformly Lipschitz in \( B(\frac{1 + r}{2}) - D_\frac{r}{2}(\frac{1 + r}{2}) \) for \( r \in (0, 1) \). First, we wish to show that
\[
\lim_{k \to \infty} E^{h_k}[B(r) - D_\epsilon(r)] = E^{\ast}[B(r) - D_\epsilon(r)].
\]
This follows essentially from [KS1] Theorem 3.11, expect that in order to apply this theorem, we need $B(r) - D_{\epsilon}(r)$ to be a smooth domain. Therefore, a small modification of their argument is needed which is given below. By [KS1] Theorem 3.11, the energy densities of $h_k$ converge to those of $f_*$ in any smooth domain $\Omega \subset B(\frac{1+r}{2}) - D_{\frac{1+r}{2}}$, $j = 1, \ldots, N$. Let $\Omega_1, \ldots, \Omega_L$ be such domains and so that $B(r) - D_{\epsilon}(r) \subset \Omega_1 \cup \ldots \cup \Omega_L$. Since $|\nabla h_k|^2 d\mu$ converge to $|\nabla f_*|^2 d\mu$, we have that

$$E^{h_k}[\Omega'] \rightarrow E^{f_*}[\Omega']$$

for any $\Omega' \subset \Omega$. If $\Omega'_{l_0} = \Omega_{l_0} - (\Omega_1 \cup \ldots \cup \Omega_{l_0-1} \cup D_{\epsilon}(r))$ for $l_0 = 1, \ldots, L$, then

$$\lim_{k \to \infty} E^{h_k}[B(r) - D_{\epsilon}(r)] = \lim_{k \to \infty} \sum_{l_0=1}^L E^{h_k}[\Omega'_{l_0}]$$

$$= \sum_{l_0=1}^L E^{f_*}[\Omega'_{l_0}]$$

$$= E^{f_*}[B(r) - D_{\epsilon}(r)]$$

By Lemma 46,

$$E^{h_k}[D_{\epsilon}(r)] \leq C \epsilon^\delta$$

for any $\epsilon$ sufficiently small. Thus,

$$\lim_{k \to \infty} E^{h_k}(r) - C \epsilon^\delta \leq \lim_{k \to 0} E^{h_k}[B(r) - D_{\epsilon}(r)] \leq E^{f_*}[B(r) - D_{\epsilon}(r)] \leq E^{f_*}(r).$$

By lower semicontinuity of energy,

$$E^{f_*}(r) \leq \liminf_{k \to 0} E^{h_k}(r) \leq \limsup_{k \to 0} E^{h_k}(r) \leq E^{f_*}(r) + C \epsilon^\delta.$$  

Since $\epsilon > 0$ can be made arbitrarily small, this proves (70). To prove (71), we see that

$$E^{f_*}(r) = \liminf_{k \to \infty}(1 - c\lambda_k) E^{h_k}(r)$$

$$\leq \liminf_{k \to \infty} g^k E^{h_k}(r)$$

$$\leq \liminf_{k \to \infty} g^k E^{h_k}(r)$$

$$\leq \limsup_{k \to \infty}(1 + c\lambda_k) E^{h_k}(r)$$

$$= E^{f_*}(r)$$

$$= E^{f_*}(r).$$
Here, the last line follows from the fact that $f_*=h_*$ by Proposition 37. Since there is no loss of total energy, we see that the directional energies converge by using the lower semicontinuity. Q.E.D.

**Lemma 48** Let $g$ be a normalized Lipschitz metric on $B(r)$ and $f : (B(r), g) \to Y$ a harmonic map. Then its tangent map $f_* : B(1) \to Y_*$ is homogeneous of order $\alpha$ where $\alpha$ is the order of $f$ at 0.

**Proof.** Follows immediately from Lemma 38 and Lemma 47. Q.E.D.

**Lemma 49** Let $g$ be a normalized Lipschitz metric on $B(\cdot)$, $f : (B(\cdot), g) \to Y$ a harmonic map and $f_* : B(\cdot) \to Y_*$ its tangent map. For every $\beta, r \in (0, 1)$, there exists $B$ so that

$$d_*(f_*(x), f_*(y)) \leq B|x - y|^{\beta}$$

for all $x, y \in B(\cdot)$ and $B$ is only dependent on the choice of $\beta, r$ and the total energy of $f_*$. 

**Proof.** First, note that $h_k$ converges to $f_*$ uniformly by Proposition 37. Next, note that the energy of $h_k$ converges to that of $f_*$ by Lemma 47. Thus, the result follows from Lemma 45. Q.E.D.

**Lemma 50** Let $B$ be a dimension-$n$, codimension-1 local model, $g$ be a uniformly elliptic Lipschitz metric on $B(\cdot)$ and $f : (B(\cdot), g) \to Y$ be a harmonic map. Then the order $\alpha$ of $f$ at 0 is $\geq 1$.

**Proof.** Since $f_*$ is homogeneous of degree $\alpha$,

$$d_*(f_*(tx), f_*(0)) = |tx|^{\alpha} d_*(f_*\left(\frac{x}{|x|}\right), f_*(0)).$$

On the other hand, for any $\beta \in (0, 1)$ and $t$ small, there exists a constant $B$ so that

$$d_*(f_*(tx), f_*(0)) \leq B|tx|^\beta$$

by Lemma 49. Thus,

$$d_*(f_*\left(\frac{x}{|x|}\right), f_*(0)) \leq B|tx|^\beta - \alpha.$$  

If $\alpha < 1$, choose $\beta$ so that $\beta > \alpha$ and take the limit as $t \to 0$ to obtain

$$d_*(f_*\left(\frac{x}{|x|}\right), f_*(0)) = 0.$$
Since the choice of $x \in B(1)$ is arbitrary, this contradicts Lemma 35. Q.E.D.

Using the fact that the order at a point on $D$ is $\geq 1$, we can prove Lipschitz continuity in $B(1)$.

**Theorem 51** Let $B$ be a dimension-$n$, codimension-1 local model, $g$ be a uniformly elliptic Lipschitz metric on $B(r)$ and $f : (B(r), g) \to Y$ be a harmonic map. Then $f$ is Lipschitz in $B(\frac{r}{2})$ with Lipschitz constant dependent on $g$ and the total energy $E\delta$ of $f$.

**Proof.** For each $x \in B(1)$, the normalized map $f \circ L_x$ (cf. Proposition 4) has order $\geq 1$ at 0 by Lemma 50. Thus, the order $\alpha_x$ of $f$ at $x$ is $\geq 1$. The result now follows from Theorem 29. Q.E.D.

8.3 At a higher codimension singular point

Now we consider a dimension-$n$, codimension-$\nu$ local model where $\nu \geq 2$. Generally, we do not expect a harmonic map from this space to be Lipschitz continuous. On the other hand, we show that Lipschitz continuity can be proved with an additional assumption.

First, we establish some properties of the tangent map. Lemmas 52, 53 and 54 below is an analogues of Lemmas 46, 47 and 42 corresponding to the codimension-$\nu$ case for $\nu \geq 2$.

**Lemma 52** Let $B$ be a dimension-$n$, codimension-$\nu$ local model with $\nu \geq 2$, $g$ a normalized Lipschitz metric defined on $B(r)$, $(Y, d)$ a NPC space and $f : (B(r), g) \to Y$, $h_k : (B(1), \delta) \to Y$, $f_k : (B, g) \to Y$, $h_\ast = f_\ast : B(1) \to Y$ is defined as in Section 6. Let $D_\epsilon$ be the $\epsilon$-neighborhood of $D$ and $D_\epsilon(r) = B(r) \cap D_\epsilon$. Fix $R \in (0, 1)$. For any $r \in (0, R)$, there exists $C$ and $\delta > 0$, $k_0$ sufficiently large and $\epsilon_0 > 0$ sufficiently small so that

$$E[h_k[D_\epsilon(r)] \leq C\epsilon^\delta, \forall k > k_0, \epsilon < \epsilon_0.$$  

**Proof.** As in the proof of Lemma 46, there exists a constant $c_3$ so that

$$\frac{\epsilon E[h_k]}{\epsilon} \leq c_3.$$  

Thus, by Theorem 30,

$$E[h_k] \leq \frac{c_3f_k}{\epsilon} \leq \frac{c_3C^2\epsilon^{2\gamma+n-1}}{\epsilon} = c_4\epsilon^{2\gamma+n-2}.$$  

We can cover $D_\epsilon(r)$ by $\frac{c_6}{\epsilon^{\nu-\nu}}$ number of $(2\epsilon)$-balls. Thus,

$$E[h_k[D_\epsilon(r)] \leq \frac{c_6}{\epsilon^{\nu-\nu}}c_4\epsilon^{2\gamma+n-2} = c_6\epsilon^{2\gamma-2+\nu}.$$  

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The lower semicontinuity of energy implies that
\[ E_{f^*}[D_\epsilon(r)] \leq c_0 \epsilon^{2\gamma - 2 + \nu}. \]
Q.E.D.

**Lemma 53** Let \( h_k, f_k, h_* = f_* \) be as in Lemma 52.
\[ \lim_{k \to 0} E^{h_k}(r) = E^{h_*}(r). \]
and
\[ \lim_{k \to \infty} E^{f_k}(r) = E^{f_*}(r). \]
Furthermore, the directional energies of \( f_k, h_k \) converge to that of \( f_* \). The maps \( f_* \) is a homogeneous map of order \( \alpha \), where \( \alpha \) is the order of \( f \) at 0.

**Proof.** Using Lemma 52, we can follow the argument of the proof of Lemma 47. Q.E.D.

**Lemma 54** Let \( B \) be a dimension-\( n \), codimension-\( \nu \) local model, \( g \) a normalized Lipschitz metric defined on \( B(r), (Y, d) \) a NPC space, \( f : (B(r), g) \to (Y, d) \) a harmonic map and \( f_* : B(1) \to Y_* \) its tangent map. Then for \( x, y \) on the same wedge at a distance \( \rho \) away from \( D(1) \), we have
\[ d(f_*(x), f_*(y)) \leq L|x - y| \]
where \( L \) is dependent only on \( r \) and \( E^h \).

**Proof.** If \( x, y \in B(1/2) \) are a set of points on the same wedge at a distance \( \rho \) away from \( D(1) \), then Lemma 42 implies
\[ d(h_k(x), h_k(y)) \leq L|x - y| \]
where \( c \) is dependent on \( r \) and \( E^{h_k} \). Thus, the result follows from the uniform convergence of \( h_k \) to \( f_* \) and the convergence of the energy of \( h_k \) to that of \( f_* \). Q.E.D.

**Lemma 55** Let \( B \) be a dimension-\( n \), codimension-\( \nu \) local model, \( g \) a normalized Lipschitz metric defined on \( B(r), (Y, d) \) a NPC space and \( f : (B(r), g) \to (Y, d) \) a harmonic map. Then its tangent map \( f_* : B(1) \to Y_* \) is homogeneous of order \( \alpha \) where \( \alpha \) is the order of \( f \) at 0.
Our next goal is to relate the order (and hence the Hölder exponent) of a harmonic map to the first eigenvalue associated with the domain and the target space. We start with a general definition of the first eigenvalue. Let $G$ be a Riemannian complex and $T$ a NPC space. A center of mass of a map $\varphi \in L^2(G,T)$ is a point $\bar{\varphi} \in T$ so that

$$\int_G d_T^2(\varphi, \bar{\varphi})ds = \inf_{P \in Y} \int_G d_T^2(\varphi, P)ds.$$ 

The unique existence of such a point is guaranteed by the NPC condition (cf. [KS1] Proposition 2.5.4). Now let $G(T)$ be the set of Lipschitz maps $\varphi : G \to T$ into a NPC space $T$ and define the first eigenvalue of $G$ with values in $T$ as

$$\lambda_1(G, T) = \inf_{\varphi \in G(T)} \int_G |\nabla \varphi|^2 ds \int_G d^2(\varphi, \bar{\varphi})ds.$$  \hspace{1cm} (72)

In the applications, $G$ will be a spherical complex associated with the domain of the map and the NPC space $T$ will be a tangent cone of the target NPC space $Y$. The following results appear in [DM3] in the case when the domain is of dimension 2.

**Lemma 56** Suppose $f : B(r) \to Y$ is a bounded map, $\sigma \in (0, r)$ and $Q \in Y$ so that

$$\int_{\partial B(\sigma)} d^2(f, Q)ds = \inf_{P \in Y} \int_{\partial B(\sigma)} d^2(f, P)ds.$$

If $\pi : Y \to T_QY$ is the projection map into the tangent cone of $Y$ at $Q$, then

$$\int_{\partial B(\sigma)} d_{T_QY}^2(\pi \circ f, 0)ds = \inf_{V \in T_QY} \int_{\partial B(\sigma)} d_{T_QY}^2(\pi \circ f, V)ds,$$

where $0$ is the origin of $T_QY$.

**Proof.** Let $t \mapsto c(t)$ be a geodesic so that $c(0) = Q$. By the minimizing property of $c(0) = Q$, we have

$$0 \leq \int_{\partial B(\sigma)} d^2(f, c(t))ds - \int_{\partial B(\sigma)} d^2(f, c(0))ds.$$

Furthermore, by Bridson-Haeflinger, Corollary II 3.6, we have

$$\lim_{t \to 0} \frac{d(f, c(t)) - d(f, c(0))}{t} = -\cos \angle(c, \gamma)$$
where $\gamma_y$ is the geodesic from $c(0)$ to $f(y)$ and $(\gamma_y, c)$ is the angle between $\gamma_y$ and $c$ at $c(0) = Q$. Therefore,

$$0 \leq \lim_{t \to 0} \int_{\partial B(\sigma)} \frac{d^2(f,c(t)) - d^2(f,c(0))}{t} dt = \lim_{t \to 0} \int_{\partial B(\sigma)} \frac{d(f,c(t)) - d(f,c(0))}{t}(d(f,c(t)) + d(f,c(0)))ds = -2 \int_{y \in \partial B(\sigma)} \cos \angle(\gamma_y, c)d(f(y), c(0))ds.$$

Let $[c]$ be the equivalence class of $c$ and $V = ([c], 1) \in T_Q Y$. Since $\pi \circ \gamma_y$ is the (radial) geodesic from the origin 0 and $\pi \circ f(y)$ in $T_Q Y$,

$$\cos \angle(\gamma_y, c)d(f(y), f(0)) = < \pi \circ f(y), V >,$$

and thus

$$0 \leq -\int_{y \in \partial B(\sigma)} < \pi \circ f(y), V > ds. \quad (73)$$

By the continuity of the inner product, (73) holds for all $V = (V_0, t) \in T_Q Y$ where $V_0 = V/|V|$. Therefore, for $t \geq 0$,

$$\int_{\partial B(\sigma)} d^2_{T_Q Y}(\pi \circ f(y), (V_0, t))ds = \int_{\partial B(\sigma)} t^2 + |\pi \circ f(y)|^2 - 2t < \pi \circ f(y), V_0 > ds \geq \int_{\partial B(\sigma)} |\pi \circ f(y)|^2ds = \int_{\partial B(\sigma)} d^2_{T_Q Y}(\pi \circ f(y), 0)ds.$$

Q.E.D.

**Corollary 57** Suppose $f : B(r) \to Y$ is a bounded map, $\sigma \in (0, r)$ and $Q \in Y$ so that

$$\int_{\partial B(\sigma)} d^2(f,Q)ds = \inf_{P \in Y} \int_{\partial B(\sigma)} d^2(f,P)ds.$$  

If $\pi : Y \to T_Q Y$ is the projection map into the tangent cone of $Y$ at $Q$ and $\sigma : B(1) \to B(\sigma)$ is defined by $\sigma(x) = \sigma x$, then

$$\frac{\int_{\partial B(1)} |\nabla^0(\pi \circ f \circ \sigma)(x)|^2ds}{\int_{\partial B(1)} |\pi \circ f \circ \sigma(x)|^2ds} \geq \lambda_1(\partial B(1), T_Q Y),$$

where $\nabla^0$ indicates that we are taking the tangential part of the energy density function on $\partial B(1)$.  

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Proof. By Lemma 56, the center of mass of the map $\pi \circ f \circ \sigma$ is 0. Thus, the assertion follows immediately from the definition of $\lambda_1(B(1), T_Q Y)$. Q.E.D.

A consequence of Corollary 57 is the following theorem which associates the first eigenvalue with the order of a harmonic map.

**Theorem 58** Let $B$ be a dimension-$n$, codimension-$\nu$ local model, $g$ an uniformly elliptic Lipschitz Riemannian metric defined on $B(r)$, $Y$ a NPC space and $f : (B(r), g) \rightarrow Y$ be a harmonic map. If $\lambda_1(\partial B(1), T_Q Y) \geq \beta$ ($> \beta$) for all $Q \in Y$ then $\alpha(\alpha + n - 2) \geq \beta$ ($> \beta$), where $\alpha$ is the order of $f$ at 0.

Proof. By inequalities (59), (60) and (63), it suffices to assume that the volume form and the directional derivatives are with respect to the Euclidean metric $\delta$. This assumption is clearly without any loss of generality since in this proof we are interested in rescalings of $B(\sigma)$ to unit size as $\sigma \rightarrow 0$ and at small scales the metric $g$ is approximately Euclidean. We emphasize that $f$ is harmonic with respect to the metric $g$ which is not necessarily Euclidean.

Let $\sigma_i \rightarrow 0$ so that $f_{\sigma_i} \rightarrow f_* : B(1) \rightarrow Y_*$. From Lemma 53, there exists $\lambda$ so that

$$\lim_{\sigma \rightarrow 0} \int_{\partial B(\lambda)} |\nabla^\partial f_{\sigma_i}|^2 ds = \int_{\partial B(\lambda)} |\nabla^\partial f_*|^2 ds.$$  

By [GS], pages 200-201, we have

$$\lim_{\sigma \rightarrow 0} \int_{\partial B(\sigma \lambda)} \frac{\sigma \lambda E(\sigma \lambda)}{d^2(f, Q_{\sigma \lambda})} ds = \lim_{\sigma \rightarrow 0} \frac{\sigma \lambda E(\sigma \lambda)}{\int_{\partial B(\sigma \lambda)} d^2(f, f(0)) ds},$$

where $Q_\sigma \in Y$ is the point so that

$$\int_{\partial B(\sigma \lambda)} d^2(f, Q_\sigma) ds = \inf_{Q \in Y} \int_{\partial B(\sigma \lambda)} d^2(f, Q) ds.$$  

This then implies

$$\lim_{\sigma \rightarrow 0} \frac{\int_{\partial B(\sigma \lambda)} d^2(f, Q_{\sigma \lambda}) ds}{\int_{\partial B(\sigma \lambda)} d^2(f, f(0)) ds} = 1. \quad (74)$$

Let $Q_1 := Q_\sigma \in Y$ and $\pi_1 : Y \rightarrow T_{Q_1} Y$ be a projection map into the tangent cone of $Y$ at $Q_1$. By Lemma 56,

$$\int_{\partial B(\sigma \lambda)} d^2(\pi_1 \circ f, 0) ds = \inf_{V \in T_{Q_1} Y} \int_{\partial B(\lambda \sigma_1)} d^2(\pi_1 \circ f, V) ds. \quad (75)$$

Additionally,

$$d^2(f, Q_1) = |\pi_1 \circ f|^2 \text{ and } |\nabla^\partial f|^2 \geq |\nabla^\partial (\pi_1 \circ f)|^2 \quad (76)$$
since $\pi_1$ is distance non-increasing. Thus, by (74) and (76),

$$
\lim_{\sigma_i \to 0} \lambda^2 \int_{\partial B(\lambda)} |\nabla^0 f_\sigma|^2 ds = \lim_{\sigma_i \to 0} \frac{\lambda^2 \int_{\partial B(\sigma, \lambda)} |\nabla^0 f|^2 ds}{\int_{\partial B(\sigma, \lambda)} d^2(f, f(0)) ds}
$$

$$
= \lim_{\sigma_i \to 0} \frac{(\sigma, \lambda)^2 \int_{\partial B(\sigma, \lambda)} |\nabla^0 f|^2 ds}{\int_{\partial B(\sigma, \lambda)} d^2(f, f(0)) ds}
$$

$$
\geq \lim_{\sigma_i \to 0} \frac{(\sigma, \lambda)^2 \int_{\partial B(\sigma, \lambda)} |\nabla^0 (\pi \circ f)|^2 ds}{\int_{\partial B(\sigma, \lambda)} |\pi \circ f|^2 ds}.
$$

By change of coordinates $y = \sigma \lambda x$, (75) and Corollary 57,

$$
\frac{(\sigma, \lambda)^2 \int_{\partial B(\sigma, \lambda)} |\nabla^0 (\pi \circ f)(y)|^2 ds}{\int_{\partial B(\sigma, \lambda)} |\pi \circ f(y)|^2 ds} = \frac{\int_{x \in \partial B(1)} |\nabla^0 (\pi \circ f \circ (\sigma, \lambda))(x)|^2 ds}{\int_{x \in \partial B(1)} |(\pi \circ f \circ (\sigma, \lambda))(x)|^2 ds}
$$

$$
= \frac{\int_{x \in \partial B(1)} |\nabla^0 (\pi \circ f \circ (\sigma, \lambda))^2(x) ds}{\int_{x \in \partial B(1)} |(\pi \circ f \circ (\sigma, \lambda))^2(x) ds}
$$

$$
\geq \lambda_1(\partial B(1), TQ, Y)
$$

$$
\geq \beta(> \beta).
$$

Therefore,

$$
R := \frac{\int_{\partial B(1)} |\nabla^0 f_\star|^2 ds}{\int_{\partial B(1)} d^2(f_\star, f_\star(0)) ds}
$$

$$
= \frac{\lambda^2 \int_{\partial B(\lambda)} |\nabla^0 f_\sigma|^2 ds}{\int_{\partial B(\lambda)} d^2(f_\sigma, f_\sigma(0)) ds}
$$

$$
= \lim_{\sigma_i \to 0} \frac{\lambda^2 \int_{\partial B(\lambda)} |\nabla^0 f_\sigma|^2 ds}{\int_{\partial B(\lambda)} d^2(f_\sigma, f_\sigma(0)) ds}
$$

$$
\geq \beta(> \beta).
$$

For $y \in \partial B(1)$, the homogeneity of $f_\star$ implies

$$
d(f_\star(r y), f_\star(0)) = r^\alpha d(f_\star(y), f_\star(0)),
$$

and hence

$$
E^{f_\star}(1) = \int_{y \in \partial B(1)} \int_0^1 \left( \left| \frac{\partial f_\star}{\partial r} (r y) \right|^2 + \frac{1}{r^2} |\nabla^0 f_\star(y)|^2 \right) r^{n-1} dr ds
$$

$$
= \int_{y \in \partial B(1)} \int_0^1 \left( \alpha^2 d^2(f_\star(y), f_\star(0)) + r^{2\alpha+n-3} |\nabla^0 f_\star(y)|^2 \right) dr ds
$$

$$
= \frac{\alpha^2}{2\alpha+n-2} \int_{y \in \partial B(1)} d^2(f_\star(y), f_\star(0)) ds + \frac{1}{2\alpha+n-2} \int_{y \in \partial B(1)} |\nabla^0 f_\star(y)|^2 ds.
$$
Thus,
\[ \alpha = \frac{E\gamma(1)}{I\gamma(1)} = \frac{\alpha^2}{2\alpha + n - 2} + \frac{1}{2\alpha + n - 2} R \]
and
\[ \alpha(\alpha + n - 2) = R \geq \beta(> \beta). \]
Q.E.D.

Given \( B \) and any \( x \in B(r) \), consider \( f \circ L : B'_x(r(x)) \to Y \) of Proposition 4 where \( B'_x \) is a local model associated with the point \( x \). We let
\[ \lambda_1 := \inf_{x \in B(r), Q \in Y} \lambda(\partial B'_x(1), T_Q Y). \]

**Corollary 59** Let \( B \) be a dimension-\( n \), codimension-\( \nu \) local model, \( g \) an uniformly elliptic Lipschitz Riemannian metric defined on \( B(r) \) and \( Y \) a NPC space. If \( \lambda_1 \geq \alpha(\alpha + n - 2) \) and \( f : B(r) \to Y \) is a harmonic map, then \( f \) is Hölder continuous with Hölder exponent \( \alpha \) in \( B(r/2) \).

**Proof.** For any \( x \in B(r/2) \), Theorem 58 says that the assumption \( \lambda_1 \geq \alpha(\alpha + n - 2) \) for all \( Q \in Y \) implies the order of \( f \) at \( x \) is \( \geq \alpha \). The result now follows from Theorem 29. Q.E.D.

We now give a sufficient condition implying that the order of a harmonic map is \( \geq 1 \). For each \( x \in D \), let \( N(x) \) be the \( \nu \)-plane perpendicular to \( D \) at \( x \). Note that \( |x| < 1 \) implies that \( \partial B(1) \cap N(x) \) is a spherical \( (\nu - 1) \)-complex. We first need the following lemma.

**Lemma 60** Let \( f_* : B(1) \to Y_* \) be a tangent map of a harmonic map. If the order of \( f_* \) is not equal to \( 1 \), then \( f_* \) is constant in the direction parallel to \( D \).

**Proof.** By Lemma 39, we may assume that \( f_* \) maps into a cone with \( f_*(0) \) equal to the vertex. Also, we may assume by homogeneity of \( f_* \) that the domain of \( f_* \) is \( B \). Let \( x, y \in B(1) \) be points on the same wedge and same distance to \( D \) and \( x' = tx, y' = ty \). Then
\[ t^\alpha d_\gamma(f_*(x), f_*(y)) = d_\gamma(f_*(x'), f_*(y')) \leq L|x' - y'| = Ct \]
where \( L \) is the Lipschitz constant of \( f_\gamma \) and \( C \) is a constant dependent on \( L \) and on the angle between the line from \( x \) to \( 0 \) and \( y \) to \( 0 \) respectively. Thus,
\[ d_\gamma(f_*(x), f_*(y)) \leq Ct^{1-\alpha}. \]
and we are done by letting \( t \to 0 \) if \( \alpha < 1 \) or \( t \to \infty \) if \( \alpha > 1 \). Q.E.D.
Theorem 61 Let $B$ be a dimension-$n$, codimension-$\nu$ local model, $g$ an uniformly elliptic Lipschitz Riemannian metric defined on $B(r)$, $Y$ a NPC space and $f : (B(r), g) \rightarrow Y$ be a harmonic map. If $\lambda_1(\partial B(1) \cap N(0), T_0 Y) \geq \beta(> \beta)$ and $\alpha < 1$ for all $Q \in Y$, then the order $\alpha$ of $f$ at $0$ satisfies $\alpha(\alpha + \nu - 2) \geq \beta(> \beta)$.

Proof. By homogeneity of $f_*$,

$$
\alpha = \frac{E^{f_*(1)}}{T^{f_*(1)}} = \frac{\int_{\partial B(1)} |\nabla f_*|^2 dy}{\int_{\partial B(1)} d^2(f_*, 0) ds} = \frac{\int_{x \in D} \int_{y \in B(1)} |\nabla f_*|^2(1 - |x|^2)^{\frac{\nu}{2}} dy dx}{\int_{x \in D} \int_{y \in B(1) \cap N(x)} |\nabla f_*|^2(1 - |x|^2)^{\frac{\nu}{2}} dy dx}.
$$

We use the notation $\nabla^N$ to indicate the we are taking the directional energy of $f_*$ on $\partial B(1) \cap N(x)$. Using Lemma 60, $|\nabla f_*|^2 = |\nabla^N f_*|^2$ and hence

$$
\int_{y \in B(1) \cap N(x)} |\nabla^N f_*|^2(y) dy = \int_{y \in B(1) \cap N(x)} |\nabla^N f_*|^2(1 - |x|^2)^{\frac{\nu}{2}} dy.
$$

Here, the second equality follows from translation in direction parallel to $D$ and the last equality follows from the homogeneity

$$
|\nabla^N f_*|^2(y) = (1 - |x|^2)^{\alpha} |\nabla^N f_*|^2((1 - |x|^2)^{1/2}y).
$$

Now apply the change of coordinates $z = (1 - |x|^2)^{1/2}y$ and we obtain

$$
\int_{y \in B\left(\frac{1}{(1 - |x|^2)^{1/2}}\right) \cap N(0)} |\nabla^N f_*|^2((1 - |x|^2)^{1/2}y) dy = (1 - |x|^2)^{\alpha - \frac{\nu}{2}} \int_{B(1) \cap N(0)} |\nabla^N f_*|^2(z) dz
$$

since the dimension of $N(0)$ is $\nu$. Hence the numerator in (77) is

$$
\int_{x \in D} (1 - |x|^2)^{\frac{\nu + 2\alpha - 2}{2}} dx \int_{B(1) \cap N(0)} |\nabla^N f_*|^2(z) dz.
$$
Similarly, the denominator of (77) is
\[
\int_{x \in D} \frac{1}{(1 - |x|^2)^{1/2}} dx \int_{\partial B(1) \cap N(x)} d^2(f_*, 0) ds
= \int_{x \in D} \frac{1}{(1 - |x|^2)^{1/2}} dx \int_{\partial B(1) \cap N(0)} d^2(f_*, 0) ds.
\]

Thus, as in the proof of Theorem 58, we obtain
\[
\alpha = \frac{\int_{\partial B(1) \cap N(0)} |\nabla N f_*|^2 dy}{\int_{\partial B(1) \cap N(0)} d^2(f_*, 0) ds}
= \frac{\alpha^2}{2\alpha + \nu - 2} + \frac{R}{2\alpha + \nu - 2}
\]
and hence
\[
\alpha(\alpha + \nu - 2) = R \geq \beta(> \beta).
\]
Q.E.D.

Given \( B \) and any \( x \in B(r) \), consider \( f \circ L_x : B'_x(r(x)) \to Y \) of Proposition 4 where \( B'_x \) is a local model associated with the point \( x \). We let
\[
\lambda^N_1 := \inf_{x \in B(r), Q \in Y} \lambda(\partial B'_x(1) \cap N(0), T_Q Y).
\]

**Corollary 62** Let \( B \) be a dimension-\( n \), codimension-\( \nu \) local model, \( g \) an uniformly elliptic Lipschitz Riemannian metric defined on \( B(r) \) and \( Y \) a NPC space. If \( \lambda^N_1 \geq \nu - 1 \) and \( f : B(r) \to Y \) is a harmonic map, then \( f \) is Lipschitz continuous in \( B(r/2) \).

**Proof.** For any \( x \in B(r/2) \), Theorem 61 says that the assumption \( \lambda^N_1 \geq \nu - 1 \) implies that the order of \( f \) at \( x \) is \( \geq 1 \). The result now follows from Theorem 29. Q.E.D.

### 9 Main Theorem

Here, we collect the regularity results from the previous sections to summarize our main regularity theorem for Lipschitz Riemannian complexes.

**Theorem 63** Let \( B(r) \) be a ball or radius \( r \) around a point \( x \) in an admissible complex \( X \) endowed with a uniformly elliptic Lipschitz metric \( g \), \( (Y, d) \) a NPC space and \( f : (B(r), g) \to Y \) a harmonic map.

(1) If \( x \in X - X^{(n-2)} \) let \( d \) denote the distance of \( x \) to \( X^{(n-2)} \). Then for any \( d'' < \min(r, \frac{d}{2}) \), \( f \) is Lipschitz continuous in \( B(d') \) with Lipschitz constant dependent on the Lipschitz bound and the ellipticity constants of \( g \), \( E^f \) and \( d \).
(2) If \( x \in X^{(k)} - X^{(k-1)} \) for \( k = 0, \ldots, n-2 \) let \( d \) denote the distance of \( x \) to \( X^{(k-1)} \). Then, for any \( d'' < \min(r, \frac{d}{2}) \), \( f \) is Hölder continuous in \( B(d'') \) with Hölder exponent and constant dependent on the Lipschitz bound and the ellipticity constant of \( g, E' \) and \( d \). More precisely, the Hölder exponent \( \alpha \) has a lower bound given by the following: If \( \lambda^N \geq \beta(> \beta) \) then \( \alpha(\alpha + n - k - 2) \geq \beta(> \beta) \). In particular, if \( \lambda^N = n - k - 1 \), then \( f \) is Lipschitz continuous in a neighborhood of \( x \).

**Proof.** The assertion follows from Theorem 51, Theorem 61 and Corollary 62. Q.E.D.

**References**


