Math 302 – Differential Equations

======Practice Final======

Grading

1  20

2  20

3  20

4  20

5  20

6  20

7  25

8  25

9  30

10 10

Total: 200

► Your PRINTED name is: Practice Final

► Please circle your section:

(1)  T 1:30  Ames 234  Karami, Arash

(2)  T 3:00  Maryland 310  McGonagle, Matthew

(3)  Th 3:00  Hodson 316  McGonagle, Matthew

(4)  T 4:30  Maryland 114  Xiao, Ling

(5)  Th 1:30  Bloomberg 272  Karami, Arash

(6)  Th 3:00  Bloomberg 176  Xiao, Ling

► Write out and SIGN the pledge:

I pledge my honor that I have not violated
the Honor Code during this examination.

Signature:  Date:

► This is a 3-hour closed book exam. This examination booklet contains 10
problems, including one bonus problem, on 13 sheets of paper including this front
cover. Please first detach the last two pages, which contains several formulas,
and are intended for use as scrap paper.
1 (20 pts.) Multiple Choices:

For each of the following problems, several possible answers are given. Circle the letters corresponding to all the correct answers. (No details are needed.)

1. The equation \( xy' + 2y = x \) is a
   A: linear equation       B: separable equation
   C: homogeneous equation   D: exact equation

2. Near the critical point (0,0), the linear system \( \vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \vec{x} \) is
   A: stable       B: unstable       C: asymptotically stable
   D: spiral       E: node       F: saddle

3. To solve the linear system \( \vec{x}' = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} \vec{x} + \begin{pmatrix} e^t \\ t \end{pmatrix} \) using the method of undetermined coefficients, we should try particular solution \( \vec{x}_p \) of the form
   A: \( e^t \vec{a} + t \vec{b} \)       B: \( t e^t \vec{a} + \vec{b} + \vec{c} \)
   C: \( t e^t \vec{a} + e^t \vec{b} + t \vec{c} \)       D: \( t e^t \vec{a} + e^t \vec{b} + t \vec{c} + \vec{d} \)

4. Which of the following can be the general solution to a second order linear ordinary differential equation?
   A: \( y = (c_1 + c_2)x + e^x \)       B: \( y = c_1 x + c_2 e^{-x} \)
   C: \( y = e^x(c_1 + c_2x) \)       D: \( y = \cos(x + c_1) + c_2 \sin 2x \)

5. How many critical points does the system \( \begin{cases} x' = x(1 - x - y) \\ y' = y(0.75 - y - 0.5x) \end{cases} \) have?
   A: 1       B: 2       C: 3       D: 4
Consider the equation
\[ \frac{dy}{dx} = \begin{cases} 
\frac{x^2}{3} & \text{when } x \geq y \\
\frac{y^2}{3} & \text{when } x \leq y 
\end{cases} \]

1. Sketch the direction field for $0 \leq x, y \leq 4$, and sketch the trajectory starting at $(0,0)$.

2. Find the solution of this equation with initial condition $y(0) = 0$ analytically.

3. At which value of $x$ will this solution “blow-up” to $\infty$?
3  (20 pts.)  Given the fact that the equation

\[ ydx + (2x - ye^y)dy = 0 \]

has an integrating factor of the type \( \mu = \mu(y) \), find out this factor and then solve this equation.
4 (20 pts.) Solve the second order differential equation

\[ y'' + 3y' + 2y = 1 \]

with boundary conditions \( y(0) = y(1) = 2 \).
We want to solve the second order linear equation

\[ xy'' - (2x + 1)y' + (x + 1)y = 0. \]

1. Show that \( y = e^x \) is a solution to this equation.

2. Use the previous fact to find the general solution.
Using Laplace transform method to solve

\[
\begin{cases}
\frac{dx}{dt} = -2x + y \\
\frac{dy}{dt} = x - 2y
\end{cases}
\]

with initial conditions \( x(0) = 3, y(0) = 1. \)
7 (25 pts.) Let \( A = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix} \).

1. Solve the linear system \( \vec{x}' = A\vec{x} \).

2. Solve the nonhomogeneous linear system \( \vec{x}' = A\vec{x} + \begin{pmatrix} t \\ 1 \end{pmatrix} \).
Let \( A = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \).

1. Find all the eigenvalues and corresponding eigenvectors of \( A \).

2. Draw the phase picture of this system near origin.
Consider the system
\[
\begin{align*}
    x' &= y \\
    y' &= x(x - 2) - y
\end{align*}
\]

1. Locate all critical points of this system.

2. Classify these critical points by the linearization method.

3. Sketch some of its global trajectories.
10 (10 pts.)  (This is only a bonus problem. Do other problems first!)

Solve \( yy'' + (y')^2 - 2yy' = 0. \)
This page is intended for use as scrap paper.

More scrap papers are available upon request.
Useful Formulas

Four your convenience, please DETACH this page before the Exam

Variation of parameter formula:

\[ Y(x) = -y_1(x) \int_0^x \frac{y_2(t)g(t)}{W(y_1, y_2)(t)} dt + y_2(x) \int_0^x \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} dt. \]

Laplace Transforms:

\[
\begin{align*}
F(s) & = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t) dt \\
\mathcal{L}[af(t) + bg(t)] & = aF(s) + bG(s) \\
\mathcal{L}[f'(t)] & = sF(s) - f(0) \\
\mathcal{L}[f''(t)] & = s^2F(s) - sf(0) - f'(0) \\
\mathcal{L}[\int_0^t f(\tau)d\tau] & = \frac{F(s)}{s} \\
\mathcal{L}[e^{at}f(t)] & = F(s-a) \\
\mathcal{L}[u_c(t)f(t-c)] & = e^{-cs}F(s) \\
\mathcal{L}[tf(t)] & = -F'(s) \\
\mathcal{L}\left[\frac{f(t)}{t}\right] & = \int_s^\infty F(\sigma)d\sigma \\
(f * g)(t) & = \int_0^t f(\tau)g(t-\tau)d\tau = (g * f)(t) \\
\mathcal{L}[f * g] & = F(s)G(s) \\
\mathcal{L}[u_0(t)] & = \frac{1}{s} \\
\mathcal{L}[u_a(t)] & = \frac{e^{-as}}{s}, a \geq 0 \\
\mathcal{L}[\delta(t-a)] & = e^{-as} \\
\end{align*}
\]

Linear Algebra:

Suppose \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is a 2 \( \times \) 2 matrix. Then

The determinant \( \text{det}(A) = ad - bc \), the trace \( \text{tr}(A) = a + d \)

The characteristic equation \( \lambda^2 - \text{tr}(A)\lambda + \text{det}(A) = \lambda^2 - (a + d)\lambda + ad - bc = 0 \).

The eigenvector \( \vec{v} \) for \( \lambda \): \( (A - \lambda I)\vec{v} = 0 \).

The inverse matrix \( A^{-1} = \frac{1}{\text{det}(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \)

The exponential matrix \( e^A = I + A + \frac{1}{2!}A^2 + \cdots + \frac{1}{m!}A^m + \cdots \).