4) \( S(t) \): amount of salt in the tank at time \( t \)

\[ S_0 = 100 \text{ lb}, \ V_1 = \text{rate of entering water}, \ V_2 = \text{rate of flowing out water} \]

\[ S(t) = S_0 + V_1 x t \times t = 100 + 3x t \]

we set \( W(t) = \text{amount of water in tank} \)

\[ W(t) = W_0 + V_1 x t = 200 + 3t \]

if \( W(t) = 500 \) \( \implies 500 = 200 + 3t \) \( \implies t = 100 \)

Time of flow out = \( t = 100 \)

So \( S(t) = 100 + 3x t \) \( \implies S(100) = 400 \text{ lb} \)

\[ W(100) = 500 \text{ gal} \]

\( \text{Concentration: } \frac{400}{500} = 0.8 \text{ (lb/gal)} \)

Limit \( \frac{S(t)}{W(t)} = \lim_{t \to \infty} \frac{100 + 7t}{t} = 0 \) \( \implies S(t) \sim 100 \)

13) Let \( P(t) \) be the population size of mosquitoes at any time.

The rate of increase of mosquito population is \( \frac{dP}{dt} \).

Population decreases by 20,000 per day.

So \( \frac{dP}{dt} = P - 20,000 = \implies P(t) = P_0 e^{rt} - \frac{20,000}{r} \left( e^{rt} - 1 \right) \)

in the absence of predators, governing equation is

\( \frac{dP}{dt} = (P_1 - P_0) \sqrt{t} \)

\( P(7) = 2P_0 = 2P_0 \implies P(7) = 2P_0 \implies P_0 \frac{\ln(2)}{7} \)

\( y = \frac{\ln(2)}{7} \implies P(t) = 2x10^5 e^\left( \frac{ln(2)}{7} \right) - 201,997(e^{(x-1)}) \)
2.4.2: Rewrite differential equation as

\[
y' + \frac{1}{t(t+4)} y = 0 \quad \text{so} \quad \frac{1}{t(t+4)} \text{ is continuous everywhere except at } t=0,4,
\]

at \( t=0,4 \), since initial condition is specified at \( t=2 \).

Theorem 2.4.1 assures the existence of a unique solution on the interval \((a, 4]\).

9) \( f(t, y) \) is discontinuous along the coordinate axes, and an hyperbola \( t^2 - y^2 = 1 \). Furthermore,

\[
\frac{\partial f}{\partial y} = \frac{1}{y(1+t^2+y^2)} - 2 \frac{y |y|}{(1-t^2-y^2)}
\]

has the same point of discontinuity.

13) \( y' = \frac{-4t}{y} = 0 \Rightarrow y^2 = -2t^2 + C \)

\( = 0, y(0) = y_0 \Rightarrow y^2_0 = C \)

\( = 0, y^2 = -4t^2 + y_0^2 \Rightarrow 4t^2 \leq y_0^2 \Rightarrow \frac{12+1}{y_0^2} \leq \frac{|y_1|}{2} \)
23) $\Phi(t) = e^t \quad \Phi'(t) = 2e^t = 2\Phi(t) \quad \Rightarrow \quad \Phi''(t) - 2\Phi(t) = 0$

$(C\Phi(t))' - 2(C\Phi(t)) = C\Phi'(t) = C(2\Phi(t)) = 0$

$\frac{d}{dt}(C\Phi(t)) = C\Phi'(t) = -C\Phi^2(t)$

27) a) $y' + p(t)y = q(t) \quad y = \frac{1}{p(t)} \int \left[ \frac{p(t)}{y} q(t) \right] dt + C$

$y' + p(t)y = q(t) \quad y = Ce$

b) $V = y^{-w}, \quad V' = y'y^{-w-1}$

$y'e^{-p(t)y} = q(t)y^{-w} \quad V' - p(t)V = q(t)$

$= 0 \quad V' + p(t)V = q(t) \quad V = (1-w) q(t)$

Now you can solve the problem

32) $y' + 2y = 1 \quad y(0) = -\infty < \infty$

$= 1 \quad \Phi(t) = e^t \quad y(t) = e^{2t} \left[ \int e^{2t} + C \right] = \frac{1}{2}e^{2t} + Ce^{2t}$

$y(0) = -1 \quad \Rightarrow \quad y(0) = \frac{1}{2} - \frac{1}{2}e^{2t} \quad y(t) = \frac{1}{2}(1-e^{2t})$

$y' + 2y = 0 \quad y(t) = \frac{1}{2}(1-e^{2t}) \quad \Rightarrow \quad !$
\[ y = C e^{-2t} \]
\[ \frac{1}{2} (1 - e^{-2t}) = C e^{-2t} = \frac{1}{2} e^{2t-1} \]
\[ y(t) = \begin{cases} 
\frac{1}{2} - \frac{1}{2} e^{-2t} & t < 1 \\
\frac{1}{2} (e^{-1} - e^{-2t}) & t \geq 1 
\end{cases} \]

\[ y = 0, \ y = 2 \] are unstable.

\[ y = 1 \] is asymptotically stable.

7. a) \[ \frac{dy}{dt} = k (1-y) y \]

1. Just \( J = 1 \) is equilibrium solution.

\[ y = \frac{k}{(1-y)^2} = -\frac{(1-y)}{3} = k \Rightarrow (y-1)^2 = C \]

\[ \Rightarrow \frac{(1-y)^3}{3} = t + \frac{(y-1)^3}{3} \]
The equilibrium solution \( y = -1 \)
is asymptotically stable, \( y = 0 \) is semistable, \( y = 1 \) is unstable.

15) Inverting Eq. (11) shows \( t \) as a function of the population \( y \) and the carrying capacity \( k \), with \( y_0 = k/8 \)

\[
t = -\frac{1}{r} \ln \left( \frac{y_3 \left[ 1 - \left( \frac{y_0}{k} \right) \right]}{(y_0) \left[ 1 - y_3 \right]} \right), \quad y = 2y_0.
\]

\[
T = -\frac{1}{r} \ln \left( \frac{y_3 \left[ 1 - \frac{y_0}{k} \right]}{(y_0) \left[ 1 - y_3 \right]} \right) \quad \text{That is } T = \frac{\ln y_3}{r} \quad \text{if } r = 0.025
\]

Per year, \( T = 55.45 \) years

b) In Eq. (13) set \( y_0/k = \alpha \) and \( y/k = \beta \) As a result

\[
T = -\frac{1}{r} \ln \left( \frac{\alpha (1 - \beta)}{\beta (1 - \alpha)} \right)
\]

Given \( \alpha = 0.1, \beta = 0.9 \) and \( r = 0.025 \) per year

\( T = 175.78 \) years.
28 \) \[ \frac{dx}{dt} = \alpha (p-x) \left( q-x \right), \quad x(0) = \]
\[ \int_{\alpha(p-x)(q-x)}^{dt} dt = \int \frac{1}{\alpha} \cdot \frac{1}{p-x} \left[ \frac{1}{p-x} - \frac{1}{q-x} \right] = t \]
\[ \frac{1}{\alpha(q-p)} \left. \frac{q-x}{p-x} \right| = t + C \]
\[ \left| \frac{q-x}{p-x} \right| = C e^{\alpha(q-p)t} \quad \left| \frac{q}{p} \right| = C e^{\alpha q t}, \quad C = \left| \frac{q}{p} \right| \]
\[ \left| \frac{q-x}{p-x} \right| = \left| \frac{q}{p} \right| e^{\alpha(q-p)t} \]
\[ \lim_{t \to \infty} x(t) = \left| \frac{q}{p} \right| \] if \( q > p \)
\[ \lim_{t \to \infty} x(t) = \left| \frac{q}{p} \right| \] if \( q < p \)

2b \) \[ \frac{dx}{dt} = \alpha (p-x)^2 \]
\[ \int \frac{dx}{(p-x)^2} = \int \frac{dt}{\alpha^2} \left( 1 + \frac{1}{p-x} \right) = \alpha t + C \]
\[ \frac{1}{p-x} = C \quad C = \frac{1}{\alpha} \quad \frac{1}{p-x} = \alpha t + \frac{1}{p} \]
\[ \lim_{t \to \infty} x(t) = \infty \quad \lim_{t \to \infty} \left| \frac{x}{p} \right| = x \to p \]
4) First divide both sides by \((2xy + 2)\)

\[
1 \cdot M(x, y) = 2xy, \quad N(x, y) = x, \quad \text{since} \quad M_x = N_y = 0
\]

The resulting equation is exact.

Integrating \(M\) with respect to \(x\), while holding \(y\) constant, results in

\[
\Psi(x, y) = xy + h(y) = y + h'(y) = y
\]

\[
h'(y) = 0 = h(y) = C
\]

Therefore the solution is defined implicitly as \(xy = C\).

Note that if \(xy + 1 = 0\) the equation is trivially satisfied.

7) \(M_y = e^{xg(y)} - 2y \sin x, \quad N_x = e^{xg(y)} - 2 \sin(x)\)

\[1 \cdot M_y = N_x = \text{exact}\]

\[
\Psi = \int (e^{x \sin y} - 2y \sin x) \, dx + h(y)
\]

\[
= e^{x \sin y} + 2y \cos x + h(y) = \Psi(x, y)
\]

\[\frac{\partial \Psi}{\partial y} = e^{x \cos y} + 2 \cos(x) + h'(y) = e^{x \cos y} + 2 \cos(x)
\]

\[h'(y) = 0 = h(y) = C
\]

\[\Psi(x, y) = e^{x \sin y} + 2y \cos x + C\]
\[ M = x^2 + xy \] \
\[ N = y^2 + y^2 \] \
\[ M_y = 2x + y, \quad N_x = 2x \] \
\[ M_N - N_M = y - y = 0 \]

\[ \psi(x, y) = \int (x^2 + x^2y) \, dx + h(y) \]
\[ \frac{\partial \psi}{\partial y} = x^2 + x^2y \]
\[ h'(y) = 0 \]
\[ h(y) = C \]
\[ \psi(x, y) = \frac{x^2y^2}{2} + x^3y = C \] is solution

22. \[ M = (x + 2) \sin y, \quad N = x \cos y \]
\[ M_y = (x + 2) \cos y, \quad N_x = \cos y \]
\[ M_N - N_M = x \] is not exact

but if we use integrating factor \( \mu(x, y) = x e^x \) then we will have
\[ M_1 = (x^2 + 2x) e^x \sin y, \quad N_1 = x^2 \cos y e^x \]
\[ \frac{\partial M_1}{\partial y} = (x^2 + 2x) e^x \cos y + 2x \sin y e^x + x^2 \cos y e^x = (x^2 + 2x) e^x \]
\[ \mu \frac{N_1}{\partial x} = 2x \cos y e^x + x^2 \cos y e^x = (x^2 + 2x) e^x \quad \text{by the integrating factor} \]
22) \( \frac{\partial \psi}{\partial y} = \psi(x, y) = \int x^2 e^x g(y) + h(x) \)

\( \psi(x, y) = x^2 e^x \sin(y) + h(x) \)

\( \frac{\partial \psi}{\partial x} = (2x + x^2)e^x \sin(y) + h'(x) = (2x + x^2)e^x \sin(y) \)

\( h'(x) = -1 \hfill h(x) = \text{constant} \)

\( \psi(x, y) = x^2 e^x \sin(y) + C_0 \)

Solution is \( \psi(x, y) = C \)

23) We have to show that \( \frac{\partial (M/M)}{\partial y} = \frac{\partial (N/M)}{\partial x} \)

\( \frac{\partial (M/M)}{\partial y} = \int Q(y) dy \quad \frac{\partial (N/M)}{\partial x} = \int P(x) dx \)

\( \frac{\partial (M/M)}{\partial y} = \int Q(y) dy \quad \frac{\partial (N/M)}{\partial x} = \int P(x) dx \)

\( M(x, y) = \mu(x) \quad N(y) = \mu(y) \)

\( \mu N_x = \mu y + Q(x) \quad \mu N_x = \mu (y) (M_y + Q(y) x) \)

\( \mu N_x = \mu y + Q(x) \quad \mu N_x = \mu (y) (M_y + Q(y) x) \)

\( \text{but } N_x = \mu y + Q(x) = \mu (y) (M_y + Q(y) x) \)

\( \frac{\partial (M/M)}{\partial y} = \frac{\partial (N/M)}{\partial x} \)\mu \text{ is integrating factor}
(4.8) The equation is not exact, since 
\[ Nx - My = 2y - 1 \] . However, 
\[ \frac{Nx - My}{M} = \frac{2y - 1}{y} \] is a function of \( y \) alone. Hence there exists \( \mu = \mu(y) \), which is a solution of \( \mu' = \left( 2 - \frac{1}{y} \right) \mu \). The latter equation is separable with \( \frac{d\mu}{\mu} = 2 - \frac{1}{y} \), one solution is \( \mu(y) = e^{2y - \ln y} = \frac{e^{2y}}{y} \).

Now rewrite the given ODE as

\[ e^{2y} dx + (2xe^{2y} - \frac{1}{y}) dy = 0 \] . Thus equation is exact

and it is easy to see that

\[ \psi(x, y) = xe^{2y} - \ln y \]

Therefore soluation of the given equation is defined implicitly

\[ \psi(x, y) = C \]

32) Multiplying both sides of ODE by \( \mu = \left[ xy(2xy) \right]^{-1} \), so we will have

\[ \left[ \frac{2}{x} + \frac{2}{2xy} \right] dx + \left[ \frac{1}{y} + \frac{1}{2xy} \right] dy = 0 \]

So \( N_x = M_y \) now

\[ \psi(x, y) = 2\ln|x| + \ln|2xy| + \ln|y| + h(y) \]

\[ = \psi(x, y) \frac{1}{2xy} + h(y), \text{ setting } y'(y) = y = \frac{1}{y} \]

\[ = \frac{1}{2xy} + h(y) = \ln|y| = \psi(x, y) = 2\ln|x| + \ln|2xy| + \ln|y| \]

\[ \Rightarrow \psi(x, y) = C \]

\[ \Rightarrow \ln(2x^2 + x^2y^2) = C \]

\[ \Rightarrow 2x^2y + x^2y^2 = C \]