SOLUTION FOR MATH417 MIDTERM

Problem 1.
(1). Since \( f(x) = 2x \) for \( 0 < x < \pi \),

\[
f(x) \sim \sum_{n \geq 1} A_n \sin(nx).
\]

Then by definition, we have

\[
A_n = \frac{2}{\pi} \int_0^\pi 2x \sin(nx) \, dx
\]

\[
= \frac{4}{n\pi} \cos(nx)x\bigg|_0^\pi + \frac{4}{n\pi} \cos(nx)dx
\]

\[
= \frac{-4}{n} (-1)^n + \frac{4}{n^2\pi} \sin(nx)\bigg|_0^\pi
\]

\[
= (-1)^{n+1} \frac{4}{n}.
\]

(2). Let \( \hat{f} \) be the odd and \( 2\pi \)-periodic extension of \( f \). By convergence theorem, the Fourier sine series \( \mathcal{S}[f] \) of \( f \) equals to \( \hat{f} \) at the point of continuity, and \( \frac{\hat{f}(x)-f(x)}{2} \) at the point of discontinuity. In conclusion, we know that for

\[
\mathcal{S}[f](x) = \begin{cases} 
0 & x = -3\pi \\
2(x + 2\pi) & -3\pi < x < -\pi \\
0 & x = -\pi \\
2x & -\pi < x < \pi \\
0 & x = \pi \\
2(x - 2\pi) & \pi < x < 3\pi \\
0 & x = 3\pi
\end{cases}
\]

(3). If we use partial sum of eigenfunctions to approximate a function, we know that we have Gibbs phenomenon (roughly 9% overshoot) at the point of discontinuity.

Here the only points of discontinuity are \( x_k = \pi + 2k\pi \), at \( x_k \), \( f(x_k+) - f(x_k-) = 4\pi \), the overshoot is roughly \( 9\% \times 4\pi = 0.36\pi \).

(4). If we studied the Fourier cosine series instead, we need to consider \( \hat{f} \) which is the even and \( 2\pi \)-periodic extension of \( f \). We can know that \( \hat{f} \) is in fact a continuous function, then by convergence theorem, the Fourier cosine series \( \mathcal{C}[f](x) \) of \( f \) equals to \( \hat{f} \).

Problem 2. By the method of separation of variables, we assume first that

\[
u(x, t) = \phi(x)h(t).
\]

Then by the equation, we know that

\[
\phi(x) \frac{\partial h}{\partial t}(t) = \frac{\partial^2 \phi}{\partial x^2}(x)h(t),
\]
and hence by dividing $\phi h$, we can separate the variables (and get two equations for $\phi$ and $h$),

\[
\frac{dh}{dt}(t)/h(t) = \frac{d^2\phi}{dx^2}(x)/\phi(x) = -\lambda.
\]

We study first the eigenvalue problem for $\phi$. Recall that we must have

\[
\phi(0) = \phi(\pi) = 0
\]

from the boundary condition if we consider the nontrivial solution. The solution for this eigenvalue problem is that

\[
\lambda_n = n^2, \quad \phi_n(x) = \sin(nx), \quad n \geq 1.
\]

For any $\lambda_n$, the solution for $h(t)$ is

\[
h_n(t) = e^{-n^2 t}.
\]

Then by the superposition principle, we claim the general solution to this problem is given by

\[
u(x, t) = \sum_{n \geq 1} A_n e^{-n^2 t} \sin(nx).
\]

(By continuity and BC, we conclude that we have identity for the solution $u = S[u(t)](x)$.)

To solve this problem, we need to check the initial data $u(x, 0) = 2x$, thus by setting $t = 0$, we are reduced to determine the coefficients $A_n$ of the Fourier sine series of $f(x) = 2x$. And now continue as in Problem 1, above, i.e.,

\[
u(x, t) = \sum_{n \geq 1} (-1)^{n+1} \frac{4 \sin(nx)}{n} e^{-n^2 t}.
\]

**Problem 3.** We use the method of eigenfunction expansion to solve the problem.

**First method (combine the method of shifting the data)** We have inhomogeneous BC, we first construct one reference function $r(x, t)$ s.t.

\[r_x(0, t) = 0, \quad r_x(\pi, t) = 2\pi\]

we choose $r(x, t) = x^2$.

Let $v(x, t) = u(x, t) - r(x, t)$, the the problem satisfied by $v$ is

\[
\begin{cases}
\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \cos x + 2 & (0 < x < \pi, t > 0), \\
\frac{\partial v}{\partial x}(0, t) = 0, \quad \frac{\partial v}{\partial x}(\pi, t) = 0 & (t > 0), \\
v(x, 0) = 1 & (0 < x < \pi).
\end{cases}
\]

Now we use the method of eigenfunction expansion. Recall the BC is of second type and the corresponding eigenfunctions are

\[
\lambda_n = n^2, \quad \phi_n(x) = \cos nx
\]

If $v$ solves the problem, we have for some $a_n(t)$ (we have identity due to the continuity)

\[
v(x, t) = a_0(t) + \sum_{n \geq 1} a_n(t) \cos nx
\]

We use the term by term differentiation to conclude that

\[
v_t(x, t) = a'_0(t) + \sum_{n \geq 1} a'_n(t) \cos(nx),
\]
\[ v_x(x,t) = - \sum_{n \geq 1} na_n(t) \sin(nx), \]

By BC for \( v \), we have

\[ v_{xx}(x,t) = - \sum_{n \geq 1} n^2 a_n(t) \cos(nx). \]

Then the PDE tells us that

\[ a_0'(t) + \sum_{n \geq 1} (a_n'(t) + n^2 a_n(t)) \cos nx = 2 + \cos x \]

that is

\[ a_0'(t) = 2, \ a_1'(t) + a_1(t) = 1, \ a_n'(t) + n^2 a_n(t) = 0, \ n \geq 2 \]

Moreover, IC tells us that

\[ a_0(0) = 1, \ a_1(0) = 0, \ a_n(0) = 0, \ n \geq 2 \]

Solving the ODEs for \( a_n \), we know that

\[ a_0(t) = 1 + 2t, \ a_1(t) = 1 - e^{-t}, \ a_n(t) = 0 \]

And so

\[ v(x,t) = 1 + 2t + \cos x - e^{-t} \cos x \]

Recall definition of \( v \), we have

\[ u(x,t) = 1 + 2t + (1 - e^{-t}) \cos x + x^2. \]

**Second method (direct use of method of eigenfunction expansion)**

First, note that if we consider the case that the boundary condition is homogeneous, then the eigenfunction for the eigenvalue problem

\[ \phi_{xx} + \lambda^2 \phi = 0, \quad \phi_x(0) = \phi_x(\pi) = 0, \]

will be given by

\[ \lambda_n = n^2, \quad \phi_n(x) = \cos(nx), \quad n = 0, 1, 2, \cdots. \]

Then we assume that the solution \( u \) take the form

\[ u(x,t) = \sum_{n \geq 0} A_n(t) \cos(nx). \]

We use the term by term differentiation to conclude that

\[ u_t(x,t) = \sum_{n \geq 0} A_n'(t) \cos(nx), \]

\[ u_x(x,t) = - \sum_{n \geq 1} nA_n(t) \sin(nx). \]

Moreover, by the boundary condition, we have

\[ u_{xx}(x,t) = - \sum_{n \geq 1} n^2 A_n(t) \cos(nx) + \frac{1}{\pi} u_x |_{0}^{\pi} + \sum_{n \geq 1} \frac{2}{\pi} u_x(y,t) \cos(ny) |_{0}^{\pi} \cos(nx), \]

\[ = 2 + \sum_{n \geq 1} (4 \cos n\pi - n^2 A_n(t)) \cos(nx). \]

Then if \( u \) is the solution to the PDE, we must have

\[ A_0'(t) = 2, \]

\[ A_1'(t) = 1 - 4 - A_1(t), \]
$A'_n(t) = 4(-1)^n - n^2 A_n(t), \quad n \geq 2.$

To solve these ODEs, we need also to consider the initial conditions, the initial conditions for the coefficients are

$A_0(0) = a_0 + 1, \quad A_n(0) = a_n, \quad n \geq 2.$

(let $a_n$ be the Fourier coefficients of $x^2$)

Then the solution for the ODEs for the coefficients are

$A_0(t) = a_0 + 1 + 2t,$

$A_1(t) = a_1 e^{-t} + 3e^{-t} - 3 = (a_1 + 3)(e^{-t} - 1) + a_1,$

$A_n(t) = a_n e^{-n^2 t} + (-1)^{n+1} \frac{4}{n^2}(e^{-n^2 t} - 1) = (a_n + (-1)^n \frac{4}{n^2})(e^{-n^2 t} - 1) + a_n, \quad n \geq 2.$

In conclusion, the solution is

$u(x, t) = A_0(t) + \sum_{n \geq 1} A_n(t) \cos(nx).$

Here, we can observe that we have in fact got the same result, since $a_n = (-1)^n \frac{4}{n^2}$ for $n \geq 1$

$A_0(t) = a_0 + 1 + 2t,$

$A_1(t) = a_1 e^{-t} + 3e^{-t} - 3 = -(e^{-t} - 1) + a_1,$

$A_n(t) = (a_n + (-1)^n \frac{4}{n^2})(e^{-n^2 t} - 1) + a_n, \quad n \geq 2.$

$u(x, t) = A_0(t) + \sum_{n \geq 1} A_n(t) \cos(nx) = 1 + 2t + x^2 + (1 - e^{-t}) \cos x.$

**Bonus Problem.** Since

\[
\frac{\pi}{4} \sim \sum_{n \geq 1, \text{odd}} \frac{1}{n} \sin \frac{n\pi x}{L} = \sin \frac{\pi x}{L} + \frac{1}{3} \sin \frac{3\pi x}{L} + \frac{1}{5} \sin \frac{5\pi x}{L} + \cdots,
\]

by integration from 0 to $x$, we have

\[
\frac{\pi}{4} x = - \sum_{n \geq 1, \text{odd}} \frac{1}{n \pi} L \left( \cos \frac{n\pi x}{L} - 1 \right) = a_0 - \sum_{n \geq 1, \text{odd}} \frac{L}{n \pi} \cos \frac{n\pi x}{L},
\]

that is

\[
x = A_0 - \sum_{n \geq 1, \text{odd}} \frac{4L}{(n\pi)^2} \cos \frac{n\pi x}{L}.
\]

Here we see that $A_0$ is the first coefficient of the cosine series, and so

\[
A_0 = \frac{1}{L} \int_0^L x \, dx = L/2
\]

Now we conclude that

\[
x = C[x](x) = L/2 - \sum_{n \geq 1, \text{odd}} \frac{4L}{(n\pi)^2} \cos \frac{n\pi x}{L}.
\]

By take $x = 0$, we see that

\[
0 = C[x](0) = L/2 - \sum_{n \geq 1, \text{odd}} \frac{4L}{(n\pi)^2}.
\]
that is
\[ \sum_{n \geq 1, \text{odd}} \frac{1}{n^2} = \frac{\pi^2}{8}. \]

To conclude, let \( C = \sum_{n \geq 1} \frac{1}{n^2} \), we see that
\[ \sum_{n \geq 1, \text{even}} \frac{1}{n^2} = \sum_{n = 2k, k \geq 1} \frac{1}{4k^2} = \frac{1}{4} C \]
and so
\[
C = \sum_{n \geq 1} \frac{1}{n^2} = \sum_{n \geq 1, \text{odd}} \frac{1}{n^2} + \sum_{n \geq 1, \text{even}} \frac{1}{n^2} = \frac{\pi^2}{8} + \frac{1}{4} C
\]
\[ C = \frac{4}{3} \frac{\pi^2}{8} = \frac{\pi^2}{6} \]