**Name** ________________________________________ **Major** ____________________

# MATH 409 - QUIZ # 1
## March 21st 2006, Time: 60 minutes

---

This examination booklet contains 4 problems on 5 sheets of paper including the front cover. This is a closed book exam.

---

| Problem | Possible score | Your score |
|---|---|---|
| 1 | 25 | |
| 2 | 25 | |
| 3 | 25 | |
| 4 | 25 | |
| Total | 100 | |

WRITE OUT AND SIGN THE PLEDGE:

I agree to complete this exam without unauthorized assistance from any person, materials, or device.

1. Let $\{x_n\}_{n\in\mathbf{N}}$ be a sequence of real numbers.

   (a) Assume that

   $$|x_{n+1} - x_n| < \frac{1}{2^n}, \quad \forall n \geq 20.$$

   Can you conclude that the sequence converges? Prove your claim.

   (b) Assume that

   $$|x_{n+1} - x_n| < \frac{1}{n}, \quad \forall n \geq 20.$$

   Can you conclude that the sequence converges? Prove your claim.

**Proof**

**(a)** Yes, we can. Let $n, m \geq 20$, and without loss of generality assume $n > m$. The inequality $|x_{n+1} - x_n| \leq 2^{-n}$ for $n \geq 20$ shows that

$$|x_n - x_m| = \left| \sum_{k=m+1}^{n} (x_k - x_{k-1}) \right| \leq \sum_{k=m+1}^{n} |x_k - x_{k-1}|$$

$$\leq 2^{-m} \sum_{j=0}^{n-m-1} 2^{-j} \leq 2^{1-m}$$

independent of $n$. This bound shows that $x_n$ is a Cauchy sequence, and hence converges.

**(b)** No, we cannot. Define $x_n$ as

$$x_n = \sum_{j=1}^{n} \frac{1}{j}$$

Then $|x_{n+1} - x_n| = \frac{1}{n+1} \leq \frac{1}{n}$ for all $n \geq 1$. But $x_n$ diverges (see practice midterm).

2. Let $f, g$ be differentiable functions on $(a, b)$, continuous on $[a, b]$. Assume that $g'(x) \neq 0, \forall x \in (a, b)$.

   (a) Show that $g(a) \neq g(b)$.

   (b) Show that there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

   (HINT: Use the mean value theorem applied to a function of the form $\alpha f + \beta g$ for appropriate constants $\alpha$ and $\beta$.)

**Proof**

**(a)** If $g(a) = g(b)$, the mean value theorem tells us that $g'(c) = 0$ for some $c \in (a, b)$, which contradicts our hypothesis.

**(b)** Let $\alpha = g(b) - g(a)$ and $\beta = f(a) - f(b)$. Then $(\alpha f + \beta g)(a) = (\alpha f + \beta g)(b)$. By the mean value theorem, then, there is a $c \in (a, b)$ such that $\alpha f'(c) + \beta g'(c) = 0$. From what we assumed, $g'(c) \neq 0$. Thus

$$\frac{f'(c)}{g'(c)} = -\frac{\beta}{\alpha} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

3. Let $S$ be the set of limit-points of a given set $A \subset \mathbf{R}$. Prove that $S$ is closed.

**Proof**

To show that $S$ is closed, consider $S^c$, the complement of $S$. If $x \in S^c$, by the definition of limit point, there is a neighborhood $U$ of $x$ containing no points of $A$, except for possibly $x$. Thus $U$ cannot contain any points of $S$, since in any neighborhood of a limit-point of $A$, there must be an infinite number of points of $A$. Hence $S^c$ is open and $S$ is closed.

4. State whether the following claims are true (=T) or false (=F).

(a) If $f$ is strictly increasing (and differentiable), then $f'(x) > 0$ for all $x$. F

(b) If $A$ is an open set, $x \in A$ and $B$ is the set $A$ with $x$ removed, then $B$ is open. T

(c) A function which is differentiable everywhere must also be continuous. T

(d) Bounded monotone sequences converge to a real number. T

(e) A set is either open or closed. F