The midterm will be composed by 4 problems. Each problem is worth 25 points.

(1) If f is a bounded, monotone increasing, continuous function defined on the interval [a,b), then f is uniformly continuous.

Proof.
- First, we claim that f has a limit from the left at b. Let $N \in \mathbb{N}$ be sufficiently large so that $b - \frac{1}{N} \in (a, b)$ and consider the sequence $f(b - \frac{1}{N+1}), f(b - \frac{1}{N+2}), f(b - \frac{1}{N+3}), \ldots$. Since f is monotone increasing and bounded, this sequence is monotone increasing and bounded. Therefore, it has a limit, say y_0. Let $n \in \mathbb{N}$ be given. Then there exists $k_0 \in \mathbb{N}$ so that $0 \leq y_0 - f(b - \frac{1}{k}) \leq \frac{1}{n}$ for all $k \geq k_0$. Since f is monotone increasing, $0 \leq y_0 - f(x) \leq \frac{1}{n}$ whenever $x \in (b - \frac{1}{k}, b)$. Thus, $|y_0 - f(x)| \leq \frac{1}{n}$ whenever $-\frac{1}{k} < x - b < 0$. This proves our claim. (For this first part I would give some hints to guide you through the proof.)
- Next, we let $g(x)$ be defined by setting $g(x) = f(x)$ for $x \in [a, b)$ and $g(b) = y_0$. Since f is continuous on the interval $[a, b)$ and $\lim_{x \to b^-} g(x) = \lim_{x \to b^-} f(x) = y_0 = g(b)$, g is continuous on the closed interval $[a, b]$. Thus, g is uniformly continuous and hence f is also (since $f(x) = g(x)$ on $[a, b)$). □

(2) Suppose that E is a non-empty compact set. Show that $\sup(E)$ is contained in E.

Proof.
- First, note that $\sup(E)$ exists (and is finite) since E is bounded.
- Claim: For any bounded set, we always have that $\sup E$ is either in E or a limit point of E.

  Proof of claim: Suppose that $\sup(E) \notin E$. Since $\sup(E)$ is the least upper bound, for every j, we get a point $y_j \in E$ with

  $$\sup(E) - 1/j < y_j < \sup(E)\,.$$

  (Otherwise $\sup(E) - 1/j$ would be a lower upper bound.) The sequence y_j converges to $\sup(E)$, proving the claim.
- Finally, since E is closed, it contains all of its limit points.

□

(3) Suppose that x_n is a Cauchy sequence. Prove that there exists a number $N \in \mathbb{N}$ so that $|x_j| \leq N$ for every j.

Proof. Since x_n is Cauchy, there exists $M \in \mathbb{N}$ so that for all $m, n > M$ we have

$$|x_n - x_m| < 1\,.$$

In particular, if $n > M$, we have by the triangle inequality that

$$|x_n| \leq |x_{M+1}| + |x_n - x_{M+1}| < |x_{M+1}| + 1\,.$$

It follows that *every* $|x_j|$ is bounded by the maximum of

$$\{|x_1|, \ldots, |x_M|,\, |x_{M+1}| + 1\}\,.$$

(This is finite since it is the maximum of a finite set of finite numbers!) □

(4) For the 4th problem you will have 5 True/False questions.