The final will be composed by 6 problems. Each problem is worth according to its difficulty. Here is a sample final.

(1) For the 1st problem you will have 10 True/False questions covering all the material from the syllabus.

(2) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function satisfying $f(0) = 0$ and $f'(x) > f(x)$ for every $x \in \mathbb{R}$. Prove that $f(x) > 0$ for every $x > 0$.

Proof. We will argue by contradiction. Choose $x > 0$ to be the first zero of f; i.e., set

$$x = \inf\{y > 0 \,|\, f(y) = 0\}\,.$$

This inf exists because the set is bounded below by 0. Moreover, since f is cts, the set of zeros is closed and, hence, the inf is in the set — i.e., $f(x) = 0$.

Note that $x > 0$. To see this, use $f'(0) > 0$ to show that f is strictly increasing at 0.

We have arranged that $f(0) = 0$ and $f(x) = 0$, but f - and hence also f' - is positive on $(0, x)$. This gives a contradiction by the MVT: There must be some y in $(0, x)$ with

$$0 = \frac{f(x) - f(0)}{x - 0} = f'(y) > 0\,.$$

(3) Let $\{A_n\}_n$ be a monotone decreasing sequence of positive numbers converging to zero. Show that $\sum_{n=1}^{\infty}(-1)^n A_n$ is convergent. (HINT: Use that $\sum_{n=1}^{m}(-1)^n A_n = A_{m+1}\sum_{n=1}^{m}(-1)^n - \sum_{n=1}^{m}(\sum_{k=1}^{n}(-1)^k)a_n$ with $a_n = A_{n+1} - A_n$.)

(4) Let $f \in C^{n+1}(N(x_0))$. Then for every $x \in N(x_0)$ there exists x_1 between x and x_0 such that

$$f(x) - T_n(x) = \frac{f^{(n+1)}(x_1)}{(n+1)!}(x - x_0)^{n+1}.$$

(Hint given in class)

(5) If f and g are Riemann integrable on $[a, b]$, show that $\max\{f, g\}$ is Riemann integrable.

(6) Let f_n converge uniformly to f on $[a, b]$. Assume that f_n, f are integrable. Show that

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b (\lim_{n\to\infty} f_n(x))dx$$

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