

Name _____ Major _____

MATH 409 - QUIZ # 1
February 26th, Time: 50 minutes

This examination booklet contains 13 true/false questions and 2 problems on 5 sheets of paper including the front cover. This is a closed book exam.

WRITE OUT AND SIGN THE PLEDGE:

I pledge my honor that I have not violated the Honor Code during this examination.

Please circle the correct statement. No proof is necessary.

1. Let S be the set of real numbers between 0 and 1, whose decimal expansion contains only 1's and 2's. S is countable.

TRUE FALSE X

2. Define the following relation among real numbers:

$$a \sim b \quad \text{if} \quad a - b = \frac{p}{q}$$

with q even and p/q in reduced form (that is q is positive and as small as possible.) This is an equivalence relation.

TRUE FALSE X

3. Let $\{a_n\}_{n \in \mathbf{N}}$ be a sequence of non-negative real numbers converging to $a \in \mathbf{R}$. Then the sequence $\{\sqrt{a_n a_{n+1}}\}_{n \in \mathbf{N}}$ converges to a .

TRUE X FALSE

4. Every Cauchy sequence of integers converges to an integer.

TRUE X FALSE

5. All countable intersections of open intervals are open intervals.

TRUE FALSE X

6. The set $\{0, 1, 1/2, 1/3, \dots\}$ is compact.

TRUE X FALSE

7. Let $\{a_n\}_{n \in \mathbf{N}}$ be a sequence of real numbers such that

$$\lim_{n \rightarrow \infty} |a_n - a_{n+1}| = 0.$$

Then $\{a_n\}_{n \in \mathbf{N}}$ is convergent.

TRUE FALSE X

8. Let $\{a_n\}_{n \in \mathbf{N}}$ be a sequence of real numbers. Consider the subsequences $\{a_{m(n)}\}_{n \in \mathbf{N}}$ and $\{a_{j(n)}\}_{n \in \mathbf{N}}$ with $m(n) = 2n$ and $j(n) = 2n - 1$. If $\{a_{m(n)}\}_{n \in \mathbf{N}}$ and $\{a_{j(n)}\}_{n \in \mathbf{N}}$ converge to the same limit $a \in \mathbf{R}$, then $\{a_n\}_{n \in \mathbf{N}}$ converges to a .

TRUE X FALSE

9. Let E be a bounded subset of \mathbf{Q} . Then $\sup E \in \mathbf{Q}$.

TRUE FALSE X

10. There exists a sequence whose set of limit points is exactly $\{1, 1/2, 1/3, 1/4, \dots\}$.

TRUE FALSE X

11. The interval $(0, 1)$ can be written as a disjoint union of a closed and an open set.

TRUE X FALSE

12. If a countable set is removed from an uncountable set, the remainder is countable.

TRUE FALSE X

13. There exists two equivalent Cauchy sequences of rational numbers, one whose terms are all positive and the other one with all negative terms.

TRUE X FALSE

Prove the following statements.

CLAIM: The product of a negative real number and a positive real number is negative.

Proof. Let x be negative and y be positive. Then $-x$ is positive and there exists N_1, m_1 and a Cauchy sequence of rationals a_1, a_2, \dots so that $a_j \geq \frac{1}{N_1}$ for all $j \geq m_1$ (by Definition 2.2.3). Similarly, there exists N_2, m_2 and a Cauchy sequence of rationals y_1, y_2, \dots so that $a_j \geq \frac{1}{N_2}$ for all $j \geq m_2$. By Definition 2.2.2, $(-x)y$ is represented by a_1y_1, a_2y_2, \dots . If we let $m = \max\{m_1, m_2\}$, then $a_jy_j \geq \frac{1}{N_1N_2}$ whenever $j \geq m$. Thus, $(-x)y$ is a positive number. Since $xy = -(-x)y$ (by associativity), xy is negative.

CLAIM: If \mathcal{B} is an open cover of a set A , then \mathcal{B} has a countable subcover.

Proof. See the first part of Theorem 3.3.2, and notice that the assumption that A is compact is not used in the proof that \mathcal{B} admits a countable subcover.