1.2.1:

a) T b) F c) F d) F e) T f) F g) F h) F
i) T

1.2.10: We must prove: V is a subspace of the space of functions defined on $\mathbb{R}$

- $0 \in V$ (obviously)
- $f \in V \Rightarrow \forall x \in \mathbb{R}$ $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \equiv f'(x)$ exists

Need to show: For $f, g \in V$, with $K \equiv f+g$, we have

$$\lim_{h\to 0} \frac{K(x+h)-K(h)}{h} \text{ exists}$$

Pf. for $h \in \mathbb{R}$ $\frac{K(x+h)-K(h)}{h} = \frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}$

Hence, using calc I limit laws, we have

$$\lim_{h\to 0} \frac{K(x+h)-K(x)}{h} = f'(x) + g'(x)$$

i.e. $f(x) + g(x) \in V$

- Similar reasoning shows that $cf \in V$, $\forall c \in \mathbb{R}$

✓

1.2.18: The operation is Not commutative (for example)
Hence the set V is not a vector space under these operations

1.2.21: You may check that all the vector space axioms (cf pp. 7, VS 1) -> VS 8))
hold since they hold on each component.
For example, the zero element $0 \in V \times W$ under these operations is $0 = (0_V, 0_W)$
where 0_V denotes the zero element for V, and 0_W denotes the zero element for W.

1.3.11: • Denote $\mathcal{F}_{s_0} = \{f \in \mathcal{F}(S, F) \mid f(s_0) = 0\}$.

• $0 \in \mathcal{F}_{s_0}$ (obviously)

• $f, g \in \mathcal{F}_{s_0} \Rightarrow (f+g)(s_0) = f(s_0) + g(s_0) = 0$

$\Rightarrow f+g \in \mathcal{F}_{s_0}$

• $(cf)(s_0) = c f(s_0) = 0 \quad \forall c \in \mathbb{R}$

• Thus $\mathcal{F}_{s_0}$ is a vector space

1.3.9: We can write

$$W_1 \cap W_3 = \ker(A)$$

where $A = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 2 & 1 \\ 2 & -7 & 1 \end{pmatrix}$

One can compute directly that $\det(A) \neq 0$ i.e., A is invertible

$$Ax = 0 \Rightarrow x = 0$$

$\Rightarrow$ $W_1 \cap W_3 = \ker(A) = \{0\}$.

Again, we may write

$$W_1 \cap W_4 = \ker(A),$$

where

$$A = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 2 & 1 \\ 1 & -4 & -1 \end{pmatrix}$$

For the same reason, we see A is invertible and thus

$$W_1 \cap W_4 = \{0\}$$

Note that we may write

$$W_3 = \left\{ \begin{pmatrix} s \\ t \\ 7t - 2s \end{pmatrix} \,\middle|\, s, t \in \mathbb{R} \right\}$$

$$W_4 = \left\{ \begin{pmatrix} 4t + s \\ t \\ s \end{pmatrix} \,\middle|\, s, t \in \mathbb{R} \right\}$$

So for $v \in W_3 \cap W_4$, there exist $a, b, c, d \in \mathbb{R}$ s.t.

$$v = \begin{pmatrix} a \\ b \\ 7b - 2a \end{pmatrix} = \begin{pmatrix} 4c + d \\ c \\ d \end{pmatrix}$$

we see that $b = c$

$\Rightarrow$

$$a = 4b + d = 4b + 7b - 2a$$

$$\Rightarrow 3a = 11b \Rightarrow a = \tfrac{11}{3} b$$

$$\Rightarrow d = 7b - \frac{22}{3} b = \left(7 - \frac{22}{3}\right) b = \left(\frac{21}{3} - \frac{22}{3}\right) b = -\frac{1}{3} b$$

we may specify b independently.

Thus $v \in W_3 \cap W_4 \Rightarrow v = \begin{pmatrix} \frac{11}{3} t \\ t \\ t \\ -\frac{1}{3} t \end{pmatrix} = t \begin{pmatrix} \frac{11}{3} \\ 1 \\ 1 \\ -\frac{1}{3} \end{pmatrix}$

We may write

$$W_3 = \left\{ \begin{pmatrix} s \\ t \\ 7t-2s \end{pmatrix} \mid s,t \in \mathbb{R} \right\}$$

$$W_4 = \left\{ \begin{pmatrix} 4a+b \\ a \\ b \end{pmatrix} \mid a,b \in \mathbb{R} \right\}$$

Thus, for $v \in W_3 \cap W_4$, we have $\exists\, a,b,s,t \in \mathbb{R}$ s.t.

$$v = \begin{pmatrix} s \\ t \\ 7t-2s \end{pmatrix} = \begin{pmatrix} 4a+b \\ a \\ b \end{pmatrix}$$

- Immediately we have $t = a$. Thus our equations become

$$s = 4a + b \quad (1)$$

$$7a - 2s = b \quad (2)$$

- We may eliminate b form (1), (2) to get

$$s - 4a = 7a - 2s \Rightarrow \frac{11}{3}a = s \quad (3)$$

- Substituting (3) into (2), we get

$$7a - \frac{22}{3}a = b \Rightarrow b = -\frac{1}{3}a$$

1.3.19: Assume W_1, W_2 are subspaces of V w/ $W_1 \cup W_2 \equiv W$ a subspace.

Let $\alpha_i \in W_i$ $(i=1,2)$. then

$$\alpha_1 + \alpha_2 \in W \text{ by hypothesis.}$$

- $\alpha_1 + \alpha_2 \in W_1 \Rightarrow \alpha_1 + \alpha_2 = \beta$, for some $\beta \in W_1$
  $\Rightarrow \alpha_2 \in W_1$ Since $\alpha_2 = \beta - \alpha_1$
  Since α_2 was arbitrary, we get $W_2 \subseteq W_1$

- Similar reasoning shows $\alpha_1 \in W_2$ in the case $\alpha_1 + \alpha_2 \in W_2$

- We thus have either $W_1 \subset W_2$ or $W_2 \subset W_1$

1.3.23:

a) Show $W \equiv W_1 + W_2$ is a subspace of V:

- $0 \in W$ (obviously)
- Let $\alpha, \beta \in W$. Then $\alpha = \alpha_1 + \alpha_2$, $\beta = \beta_1 + \beta_2$ w/ $\alpha_i, \beta_i \in W_i$

$$\Rightarrow \quad \alpha + \beta = (\alpha_1 + \alpha_2) + (\beta_1 + \beta_2)$$

$$= \underbrace{(\alpha_1 + \beta_1)}_{\in W_1} + \underbrace{(\alpha_2 + \beta_2)}_{\in W_2}$$

Hence $\alpha + \beta \in W$

- For $c \in \mathbb{F}$, we get $c\alpha = c\alpha_1 + c\alpha_2 \in W$

Show $W_1 + W_2 \supset W_i$ $(i = 1, 2)$!

Note that $0 \in W_2$, Hence, the

Set $W_1 = \{w_1 + 0 \mid w_1 \in W_1\} \subseteq \{w_1 + w_2 \mid w_i \in W_i\}$

$= W$

Likewise $W_2 \subset W$

b) Let S' be a subspace of V with

$W_i \subset S'$.

Then, for any pair $\alpha_i \in W_i$ $(i = 1, 2)$

$\alpha_i \in S' \Rightarrow \alpha_1 + \alpha_2 \in S'$ (Since S'

is a subspace. Thus

$W_1 + W_2 \equiv \{\alpha_1 + \alpha_2 \mid \alpha_i \in W_i\} \subseteq S'$

3 b

a) Suppose $v \notin W$. Then also $-v \notin W$

$\Rightarrow v + w \neq 0$ for all $w \in W \Rightarrow$

$0 \in v + W$. ✓

Suppose $v \in W$. Then Indeed

$v + W = W$: for $w \in W$, $(w - v) \in W$

Since $v \in W$. Then the element

$v + (w - v) = w \in v + W$

Hence $v + W \supset W$. The other containment follows by Similar reasoning.

~~b) Suppose $v_1 - v_2 \in W$~~

~~Let $\alpha_1 \in v_1 + W$~~

~~Then $\alpha_1 + (v_1 - v_2) \in$~~

1.3.31:

b) Write $v_1 = v_2 + (v_1 - v_2)$. Define $\alpha = v_1 - v_2$. Assume $\alpha \in W$.

Let $\alpha_2 \in v_2 + W$. Then

$$\alpha_2 = v_2 + w \quad (w \in W)$$

$$= v_1 + (-\alpha + w)$$

$\Rightarrow \alpha_2 \in v_1 + W$, since $(w - \alpha) \in W$

So $v_2 + W \subseteq v_1 + W$. The opposite containment follows by similar reasoning.

Assume $\alpha \notin W$. Then, with α_2 as above, we get

$$\alpha_2 = v_2 + w = v_1 + (w - \alpha)$$

but $(w - \alpha) \notin W \Rightarrow v_1 + (w - \alpha) \notin v_1 + W$.

(c) Suppose ~~$v_i + W$~~

$$v_i + W = v_i' + W \quad (i = 1, 2)$$

Then

$$d_i \equiv v_i - v_i' \in W$$

Then

$$(v_1' + v_2') + W = (v_1 - d_1 + v_2 - d_2) + W$$

$$= ((v_1 + v_2) - (d_1 + d_2)) + W$$

we $d_1 + d_2 \in W$

$$\Rightarrow ((v_1 + w_2) - (d_1 + d_2)) + W$$

$$= (v_1 + v_2) + W$$

Thus, addition is well-defined. Similar reasoning shows that scalar multiplication is well-defined

d) It is easily verified that all the axioms of vector spaces are ~~verified~~ satisfied for the set

$$\{v + w \mid v \in V\}$$

over the given operations.

<u>Note</u>: Since $\{v + w \mid v \in V\}$ is not a subset of V, you cannot use the <u>subspace criteria</u>.