Hw #2 Solutions

1.4.15: Note that
$$S_1 \cap S_2 \subseteq S_1 , S_2$$

Then
$$\mathrm{Span}(S_1 \cap S_2) \subseteq \mathrm{Span}(S_1), \mathrm{Span}(S_2)$$

3 pts This is a consequence of the general fact that if
$$A \subseteq B$$
are subsets of a vector space V then
$$\mathrm{Span}(A) \subseteq \mathrm{Span}(B).$$

Pf: $\alpha \in \mathrm{Span}(A) \Rightarrow \alpha = \sum_{i=1}^{k} \alpha^i v_i$

w/ $\alpha^i \in F$, $v_i \in A$. But $A \subseteq B \Rightarrow$ $v_i \in B$ $\forall i \Rightarrow \alpha = \sum_{i=1}^{k} \alpha^i v_i \in \mathrm{Span}(B)$
Hence
$$\mathrm{Span}(A) \subseteq \mathrm{Span}(B)$$

Examples: Take $S_1 = S_2$. Then

1 pt
$$\mathrm{Span}(S_1 \cap S_2) = \underset{i=1,2}{\mathrm{Span}(S_i)} = \mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$$

Take $S_2 = cS_1$, where $0 \neq c \in F$. That is

1 pt
$$v \in S_2 \iff \tfrac{1}{c} v \in S_1$$

And assume S_1 is Linearly Independent

3 pts.

1.9.16: Let $v \in \mathrm{Span}(\beta')$ with

$$v = \sum_{i=1}^{k} \alpha^i v_i = \sum_{i=1}^{l} \beta^i w_i \quad (*)$$

Without loss of generality (w.l.o.g.), we can suppose that $v_i = w_i$ $i = 1, \dots, s$, for some $0 \leq s \leq \min(k, l)$, and ~~we can~~ that for all $i, j \geq s$, we have

$$v_i \neq w_j$$

$(*)$ may then be rewritten as

$$v = \sum_{i=1}^{s} \alpha^i v_i + \sum_{i>s} \alpha^i v_i = \sum_{i=1}^{s} \beta^i w_i + \sum_{i>s} \beta^i w_i$$

$$\Rightarrow \sum_{i=1}^{s} (\alpha^i - \beta^i) v_i + \sum_{i>s} \alpha^i v_i - \sum_{i>s} \beta^i w_i = 0$$

We get

$$\alpha^i = 0, \quad \beta^i = 0 \qquad i > s$$

$$\alpha_i - \beta_j = 0$$

1.6.24: We show that $\{f^{(k)}(x)\}_{k=0}^{n}$ is linearly independent:

Suppose

$$\sum_{k=0}^{n} c^k f^{(k)} = 0$$

- Immediately we get $c_0 = 0$, since $\deg(f^{(k)}) = n-k$
- Suppose $c_0 = 0 = \cdots c_j$

Then

$$\sum_{k=j+1}^{n} c^k f^{(k)} = 0$$

Again $c^{j+1} = 0$ since, as before $\deg(f^{(k)}) = n-k$

- Thus, by induction, we have

$$c_0 = 0 \cdots = c_n = 0$$

7 points

1.5.11: Since $\{u_1, \dots, u_n\}$ is linearly independent each linear combination

$$\sum_{i=1}^{n} \alpha_i u_i$$

corresponds to a unique element in $\mathbb{Z}$. Moreover for each $(\alpha_i)_{i=1}^{n} \in \underbrace{\mathbb{Z}_2 \times \dots \times \mathbb{Z}_2}_{n \text{ times}}$ corresponds a unique linear combination. That is, the maps

$$(\alpha_i)_{i=1}^{n} \to \sum_{i=1}^{n} \alpha_i u_i$$

is bijective. Hence

$$|\mathrm{Span}(u_1, \dots, u_n)| = |\underbrace{\mathbb{Z}_2 \times \dots \times \mathbb{Z}_2}_{n \text{ times}}| = 2^n$$

1.5.20: Suppose f, g are linearly Dependent Then $\exists\, a, b \in \mathbb{R}$ s.t.

$$af + bg \equiv 0$$

$\Rightarrow$

$$a(e^{rt}) + be^{st} = 0 \Rightarrow$$

$$a(e^{(r-s)t} + b) = 0$$

7 pts

1.5.20: Suppose

$$ae^{rt} + be^{st} = 0 \quad (\text{with } a \neq 0)$$

$\Rightarrow$

$$e^{st}\left(a e^{(r-s)t} + b\right) = 0$$

$\Rightarrow$

$$a e^{(r-s)t} + b = 0 \quad (\text{Since } e^{st} \neq 0)$$

$\Rightarrow$ $a e^{(r-s)t} = -b$

$\Rightarrow$ $r = s$, which is a contradiction, hence $a = 0$.

The same reasoning gives $b = 0$.

Hence $a = b = 0$ and f, g are linearly independent.