HW #3 Solutions to graded problems.

2.4 a) T b) F c) F d) T e) F f) F
g) T h) F

2.1.15: That T is linear is simply because integration is linear.

• <u>One to one</u>: Let $f, g \in P(\mathbb{R})$, write

$$P(\mathbb{R}) \ni f - g = \sum_{i=0}^{n} c_i x^i$$

for some $c_i \in \mathbb{R}$ $i = 1..n$, $n \in \mathbb{Z}^+$. We get

$$T(f - g) = 0$$

$$\Rightarrow \sum_{i=0}^{n} \frac{c_i}{i+1} x^{i+1} \equiv 0 \quad \forall x \in \mathbb{R}$$

$$\Rightarrow c_i \equiv 0 \quad \forall\, i = 0..n$$

$$\Rightarrow f - g \equiv 0$$

$$\Rightarrow f \equiv g$$

□

• <u>Onto (or not?)</u>: Any polynomial $f \in P(\mathbb{R})$ with

$$f = \sum_{i=0}^{n} c_i x^i, \quad c_0 \neq 0$$

is not in the image of T: given

$\sum^{k} \alpha_i x^i \in P(\mathbb{R})$ we have

$$\sum_{i=0}^{k} T(\alpha_i x^i) = \sum_{i=0}^{k} \frac{\alpha_i}{(i+1)} x^{(i+1)}$$

re-indexing we may write alternatively

$$T(\sum \alpha_i x^i) = \sum_{i=0}^{k} T(\alpha_i x^i) = \sum_{j=1}^{k+1} \frac{\alpha_{j-1}}{j} x^j$$

Thus

$$T(\sum \alpha_i x^i) = f(x)$$

$\Rightarrow$ $k = m-1$

$$\alpha_{j-1}/j = c_j \qquad j = 0 \ldots n$$

$\Rightarrow$ $$c_0 = 0$$

<u>2.2.11</u>: Let $w_1 \ldots w_k$ be a basis for the subspace W. A consequence of the Replacement Theorem (cf pp 45 Thm 1.10, pp 51 cor) The guarantees that we may extend the set $\{w_1 \ldots w_k\}$ to a basis for V. That is, we may find $v_1 \ldots v_{n-k}$ s.t.

$$\{w_1 \ldots w_k, v_1 \ldots v_{n-k}\}$$

is a basis for V.

Recall that the j^{th} column of $[T]_\beta$ is $[T(w_j)]_\beta$

But, since

$$T(w_j) \in W_1$$

We have

$$Tw_j = \alpha_j^i w_i$$

(Note that no v_i's appear in the above expression)

We thus have

$$[Tw_j]_\beta = \begin{pmatrix} \alpha_j^1 \\ \vdots \\ \alpha_j^k \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

$\square$

2.3.17: Trivially, we may write

$$x = T(x) + (x - T(x))$$

Now suppose

$$T = T^2$$

Note that

$$x - T(x) \in N(T)$$

and that for $y \in \mathrm{Im}(T)$, we have (Im(T) = R(T))

$$Ty = y$$

We have that (≤ means subspaces)

$$N(T), R(T) \leq V$$

that is, they are subspaces of V.
Moreover, for $y \in N(T) \cap R(T)$, we get

$$y = Ty = 0$$

$$\Rightarrow y = 0$$

so $N(T) \cap R(T) = \{0\}$.

On the other hand, the dimension theorem (c.f. pp 70)

Says that

$$\dim(N(T)) + \dim(R(T)) = n$$

but,
$$\dim(N(T)) + \dim(R(T)) = \dim(N(T) + R(T))$$

because $N(T) \cap R(T) = \{0\}$. We therefore get
$$N(T) \oplus R(T) = V$$

and, as we saw above we have that

$$R(T) = \{ y \mid y = Ty \}$$

Thus
$$(1) \quad T\big|_{R(T)} = id$$

$$(2) \quad T\big|_{N(T)} = 0$$

$\square$ We are done, but for completeness, we give a Matrix representation.

Let $\dim(R(T)) = K$

Then Let
$\{h_1 \dots h_K\}$ be a basis for $R(T)$.

We get
$$Th_i = h_i$$

and we get

$$T x_i = 0$$

where

$\{x_1 \dots x_{n-k}\}$ are a basis for

$K(T)$. Thus

$$[T]_\beta = \begin{array}{c|cc|c} & k & n-k & \\ \hline k & I_k & 0 & k \\ n-k & 0 & 0 & n-k \\ \hline & k & n-k & \end{array}$$