HW 4 Solutions

2.9.5. There are several ways to approach this; which you choose is dependent on what you know.

(1) A^t corresponds to the dual map between $\mathbb{R}^n$, and here must be invertible

(2) $\det(A^t) = \det(A) \neq 0$

(3) Most Directly: by definition, we have that

$$(A^{-1}A)_{ij} = \delta_{ij} \quad (\text{def of Inverse})$$

on the one hand, and on the other

$$(A^{-1}A)_{ij} = \sum_{k=1}^{n} a^{-1}_{ik} a_{kj}$$

where a^{-1}_{ij} denotes the i,jth entry of A^{-1}.
For any Matrix $B = (b_{ij})$, we have

$$(B A^T)_{ij} = \sum_{k=1}^{n} b_{ik} a^T_{kj}$$

$= \sum_{k=1}^{n} b_{ik} a_{jk}$. Now choose

$$b_{ij} = a^{-1}_{ji} \quad (\text{That is } B = (A^{-1})^T)$$

we then get

$$(BA)_{ij} = \sum_{k=1}^{n} b_{ik} a_{jk} = \sum_{k=1}^{n} a^{-1}_{ki} a_{jk} = (AA^{-1})_{ji}$$

9. Again, there are several methods

(1) $\det(AB) \neq 0 \Rightarrow \det(A), \det(B) \neq 0$, since
$\det(AB) = \det(A)\det(B)$

(2) Recall that, for a transformation

$$T: V \to V$$

on a finite dimensional vector space
Injectivity is equivalent to surjectivity. Thus

AB Invertible $\Rightarrow$ bijective $\Rightarrow \ker(AB) = 0$
$\Rightarrow \ker(B) = 0 \Rightarrow \ker(A) = 0 \Rightarrow$
A, B Injective/Surjective $\Leftrightarrow$
A, B Invertible.

Now assume $A \in Mat_{n\times m}(\mathbb{R})$, $B \in Mat_{m\times k}$
(Not necessarily square, but the product AB is defined). The previous analysis still applies partially:

- AB Invertible $\Rightarrow n = k$. Why?

$AB: \mathbb{R}^n \to \mathbb{R}^k$
$(AB)^{-1}: \mathbb{R}^k \to \mathbb{R}^n$

Now apply the dimension theorem to AB, $(AB)^{-1}$

$$\left.\begin{aligned} \dim(\mathbb{R}^n) &= \overbrace{\dim(N(AB))}^{=0} + \dim(Im(AB)) \\ \dim(\mathbb{R}^k) &= \underbrace{\dim(N((AB)^{-1}))}_{=0} + \dim(Im((AB)^{-1})) \end{aligned}\right\} =$$

Thus, we may take $n = k$. Then

$\ker(AB) = 0 \Rightarrow \ker(B) = 0 \Rightarrow B$ Injective

likewise

AB surjective $\Rightarrow$ A surjective.

Thus,

As long as B is Injective (not necessarily surjective), A is $\{$surjective$\}$ (not necessarily injective) we have

AB

is Invertible

2.5.7.

a) Note that the vectors

$$v_1 = (1, m), \quad (-m, 1) = v_2$$

are orthogonal, and that $v_1 \in L$, v_2 is perpendicular to L. Take

$$\beta = \{v_1, v_2\}$$
$$\beta' = \{e_1, e_2\}$$

We seek an expression for $[T]_{\beta'}$.
We have

$$Tv_1 = v_1$$
$$Tv_2 = -v_2$$

so

$$[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We have also

$$[I]_\beta^{\beta'}[T]_\beta = [IT]_\beta = [TI]_\beta = [T]_{\beta'}[I]_\beta^{\beta'}$$

i.e.

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q$$

where

$$Q = [I]_{\beta'}^{\beta}$$

$$Q^{-1} = [I]_{\beta}^{\beta'} = (v_1\ v_2) = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}$$

Thus

$$Q = \begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix} \cdot \frac{1}{m^2+1}$$

So

$$[T]_{\beta'} = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix}\frac{1}{m^2+1}$$

$$= \begin{pmatrix} 1 & m \\ m & -1 \end{pmatrix}\begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix} \cdot \frac{1}{m^2+1}$$

$$= \begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}\frac{1}{m^2+1}$$

$$= \begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{m^2+1} \\ 2m & \frac{m^2-1}{m^2+1} \end{pmatrix}$$

Thus

$$[Te_1]_{\beta'} = \begin{pmatrix} \frac{1-m^2}{1+m^2} \\ \frac{2m}{1+m^2} \end{pmatrix}$$

$$[Te_2]_{\beta'} = \begin{pmatrix} \frac{2m}{m^2+1} \\ \frac{m^2-1}{m^2+1} \end{pmatrix}$$

2.5.7b: Let $L^\perp$ denote the line perpendicular to L that passes through the origin. then

$$L \oplus L^\perp = \mathbb{R}^2$$

and for $\vec{x} \in \mathbb{R}^2$, we define

$$T\vec{x} = \vec{\alpha}$$

where $\vec{\alpha} \in L$, $\vec{\beta} \in L^\perp$ are unique s.t.

$$\vec{x} = \vec{\alpha} + \vec{\beta}$$

we see, with $\beta = \{v_1, v_2\}$ as in part a), that

$$Tv_1 = v_1$$
$$Tv_2 = 0$$

so

$$[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Applying again the formula

$$[T]_{\beta'} = Q^{-1} [T]_\beta Q$$

with β, Q as in part a), we get

$$[T]_{\beta'} = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix} \frac{1}{m^2+1}$$

$$= \begin{pmatrix} 1 & 0 \\ m & 0 \end{pmatrix} \begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix} \left(\frac{1}{m^2+1} \right) = \begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix} \frac{1}{m^2+1}$$

Thus, we have

$$[Te_1]_{\beta'} = \begin{pmatrix} \frac{1}{m^2+1} \\ \frac{m}{m^2+1} \end{pmatrix}$$

$$[Te_2]_{\beta'} = \begin{pmatrix} \frac{m}{m^2+1} \\ \frac{m^2}{m^2+1} \end{pmatrix}$$

2.4.2 b) Not Invertible. If

$$T: \mathbb{R}^m \to \mathbb{R}^n$$

is Invertible, then $n = m$

2.5.1 a) F b) T c) T d) F e) T

2.6.1

a) F b) T c) T d) T e) F f) T

g) T h) F