HW 1

10.1

(8) a. Use the submodule criterion. $0 \in Tor(M)$, so $Tor(M) \neq \emptyset$. Let $x, y \in Tor(M)$ and $r \in R$. Then $\exists\, s_1, s_2 \in R \setminus 0$ s.t. $s_1 x = s_2 y = 0$. $s_1 s_2 \neq 0$ since R is an integral domain and $s_1 s_2 (x + ry) = s_1 s_2 x + s_1 s_2 r y = s_2(s_1 x) + s_1 r (s_2 y) = 0 + 0 = 0$, so $x + ry \in Tor(M)$.

b. Consider $R = \mathbb{Z}/6\mathbb{Z}$ as a module over itself. Then $Tor(R) = \{0, 2, 3, 4\}$, which isn't even closed under addition.

c. Let M be a nonzero R-module and let $a, b \in R \setminus 0$ s.t. $ab = 0$. Let $m \in M$. If $bm = 0$, then $m \in Tor(M)$. If $bm \neq 0$, then $a(bm) = (ab)m = 0 \cdot m = 0$ so $bm \in Tor(M)$.

10.2

(9) Define $f: Hom_R(R, M) \to M$ by $f(\varphi) = \varphi(1)$. If $\varphi \in Hom_R(R, M)$, then $\varphi(r) = \varphi(r \cdot 1) = r \cdot \varphi(1)$, so φ is determined by $\varphi(1)$, thus f is injective. Given $m \in M$, define $\varphi(r) = r \cdot m$. Then $f(\varphi) = m$, so f is surjective. And given $r \in R$, $\varphi \in Hom_R(R, M)$ we have $f(r\varphi) = (r\varphi)(1) = \varphi(r) = r \cdot \varphi(1) = r \cdot f(\varphi)$, so f is an isomorphism of left R-modules.

10.3

(4) Let G be a finite abelian group of order n. Then $n \cdot x = 0$ for any $x \in G$ so G is a torsion $\mathbb{Z}$-module.

Now let $G = \prod_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots$, $2 \in \mathbb{Z}$ annhilates every element of G so G is an infinite torsion $\mathbb{Z}$-module.

(11) Let $\varphi : M_1 \to M_2$ be a nonzero R-module homomorphism. Then $\ker \varphi \neq M_1$ and $\operatorname{Im} \varphi \neq 0$. But $\ker \varphi$ and $\operatorname{Im} \varphi$ are submodules of M_1 and M_2 respectively so irreducibility implies $\ker \varphi = 0$ and $\operatorname{Im} \varphi = M_2$, and thus φ is an isomorphism.

If M is irreducible then the above implies that any $\varphi \neq 0$ in the ring $\operatorname{End}_R(M, M)$ has a multiplicative inverse so $\operatorname{End}_R(M, M)$ is a division ring.