HW 2

10.4

Typo: The last line on pg 375 should read "if and only if $ru'n = run'$ for some $r \in U$."

(8) a. Suppose $(u_1, n_1) \sim (u_1', n_1')$ and $(u_2, n_2) \sim (u_2', n_2')$. Then $\overline{(u_1, n_1)} + \overline{(u_2, n_2)} = \overline{(u_1u_2, u_1n_2 + u_2n_1)}$ and $\overline{(u_1', n_1')} + \overline{(u_2', n_2')} = \overline{(u_1'u_2', u_1'n_2' + u_2'n_1')}$. We have $(u_1u_2)(u_1'n_2' + u_2'n_1') = u_1u_2u_1'n_2' + u_1u_2u_2'n_1'$ $= (u_1u_1')(u_2n_2') + (u_2u_2')(u_1n_1') = (u_1u_1')(u_2'n_2) + (u_2u_2')(u_1'n_1)$ $= (u_1'u_2')(u_1n_2 + u_2n_1)$, so the two sums are equivalent. Thus addition is well-defined. Addition is clearly commutative and $\overline{(1,0)}$ is the identity element. The inverse of $\overline{(u,n)}$ is $\overline{(u,-n)}$. So $U^{-1}N$ is an abelian group. Define an R-action by $r\overline{(u,n)} = \overline{(u, rn)}$. If $(u,n) \sim (u', n')$, then $u(rn') = run' = ru'n = u'(rn)$, so $(u, rn) \sim (u', rn')$ and thus the R-action is well-defined. The module axioms clearly hold for this action, making $U^{-1}N$ an R-module

Associativity left to the reader, haha

b. Define $\varphi : Q \times N \to U^{-1}N$ by $\varphi(\frac{a}{b}, n) = \overline{(b, an)}$. If $\frac{a}{b} = \frac{c}{d}$, then $ad - bc = 0$ so $(b, an) \sim (d, cn)$ and thus φ is well-defined. We have $\varphi(\frac{a}{b} + \frac{c}{d}, n) = \varphi(\frac{ad+bc}{bd}, n) = \overline{(bd, (ad+bc)n)} = \overline{(bd, adn)} + \overline{(bd, bcn)}$ $= \overline{(b, an)} + \overline{(d, cn)} = \varphi(\frac{a}{b}, n) + \varphi(\frac{c}{d}, n)$, and $\varphi(\frac{a}{b}, n + n')$ $= \overline{(b, a(n+n'))} = \overline{(b, an)} + \overline{(b, an')} = \varphi(\frac{a}{b}, n) + \varphi(\frac{a}{b}, n')$, and $\varphi(r\frac{a}{b}, n) = \overline{(b, ran)} = \overline{(b, a(rn))} = \varphi(\frac{a}{b}, rn)$, so φ is R-balanced. Hence φ induces an R-module homomorphism $f : Q \otimes_R N \to U^{-1}N$ given by $f(\frac{a}{b} \otimes n) = \overline{(b, an)}$. Define $g : U^{-1}N \to Q \otimes_R N$ by $g(\overline{(u,n)}) = \frac{1}{u} \otimes n$. If $(u,n) \sim (u', n')$, then $\frac{1}{u} \otimes n = \frac{u'}{uu'} \otimes n = \frac{1}{uu'} \otimes u'n = \frac{1}{uu'} \otimes un' = \frac{u}{uu'} \otimes n' = \frac{1}{u'} \otimes n'$, so g is well-defined. We have $(gf)(\sum_{i=1}^{m} \frac{a_i}{b_i} \otimes n_i) = g(\sum_{i=1}^{m} \overline{(b_i, a_in_i)})$ $= g\left(\overline{\sum_{i=1}^{m} (b_1 \cdots b_m, b_1 \cdots \hat{b_i} \cdots b_m a_i n_i)}\right) = g\left(\overline{\left(b_1 \cdots b_m, \sum_{i=1}^{m} b_1 \cdots \hat{b_i} \cdots b_m a_i n_i\right)}\right)$

$$= \frac{1}{b_1\cdots b_m} \otimes \sum_{i=1}^{m} b_1\cdots\hat{b}_i\cdots b_m a_i n_i = \sum_{i=1}^{m}\left(\frac{1}{b_1\cdots b_m} \otimes b_1\cdots\hat{b}_i\cdots b_m a_i n_i\right)$$

$$= \sum_{i=1}^{m}\left(\frac{b_1\cdots\hat{b}_i\cdots b_m a_i}{b_1\cdots b_m}\otimes n_i\right) = \sum_{i=1}^{m}\frac{a_i}{b_i}\otimes n_i$$, so $gf = \mathbb{1}$. And

$(fg)(\overline{(u,n)}) = f\left(\frac{1}{u}\otimes n\right) = \overline{(u,n)}$, so $fg = \mathbb{1}$. Thus f is an R-module isomorphism of $Q\otimes_R N$ with $U^{-1}N$.

c. $\frac{1}{d}\otimes n = 0 \iff \overline{(1,0)} = f\left(\frac{1}{d}\otimes n\right) = \overline{(d,n)} \iff \exists r\in U$ s.t. $rn = d0 = 0$.

d. By (c), $1/d\otimes a = 0$ iff a is a torsion element. Thus $Q\otimes_{\mathbb{Z}} A = 0$ implies A is a torsion module. And conversely, we have $\sum \frac{b_i}{c_i}\otimes a_i = \sum\frac{1}{c_i}\otimes b_i a_i$

$= 0$.

(15) Let $M_i = \mathbb{Z}/2^i\mathbb{Z}$ for $i\geq 1$. By #8d, $Q\otimes_{\mathbb{Z}} M_i = 0$ $\forall i$. So we just need to show that $Q\otimes_{\mathbb{Z}}\left(\prod_{i=1}^{\infty} M_i\right)\neq 0$. By #8c we need to find a nontorsion element of $\prod_{i=1}^{\infty} M_i$. $(1,1,1,\dots)$ will do.

(26) First do #25: Define $S\times R[x]\to S[x]$ by $(s,f)\mapsto sf$. This induces $g: S\otimes_R R[x]\to S[x]$. Define $h: S[x]\to S\otimes_R R[x]$ by $h\left(\sum s_i x^i\right) = \sum (s_i\otimes x^i)$. Verify that g is an S-algebra homomorphism with inverse h.

By induction we get $S\otimes_R R[x_1,\dots,x_n]\cong S[x_1,\dots,x_n]$. The case $n=2$, for example, is $S\otimes_R R[x_1,x_2]\cong S\otimes_R\left(R[x_1]\otimes_R R[x_2]\right)\cong \left(S\otimes_R R[x_1]\right)\otimes_R R[x_2]$

$\cong S[x_1]\otimes_R R[x_2]\cong S[x_1][x_2]\cong S[x_1,x_2]$.

Then $S\otimes_R R[x_1,\dots,x_n]/I \cong S\otimes_R\left(R[x_1,\dots,x_n]\otimes_{R[x_1,\dots,x_n]} R[x_1,\dots,x_n]/I\right)$

$\cong \left(S\otimes_R R[x_1,\dots,x_n]\right)\otimes_{R[x_1,\dots,x_n]} R[x_1,\dots,x_n]/I \cong S[x_1,\dots,x_n]\otimes_{R[x_1,\dots,x_n]} R[x_1,\dots,x_n]/I$

$\cong S[x_1,\dots,x_n]/IS[x_1,\dots,x_n]$, where the last isomorphism is example (8) on pg 370.

The isomorphism $\varphi: S\otimes_R R[x_1,\dots,x_n]/I \to S[x_1,\dots,x_n]/IS[x_1,\dots,x_n]$ takes $s\otimes\bar{f}$ to $\overline{sf}$, and is thus far only known to be an isomorphism of modules. But

$\varphi\left(\left(\sum s_i\otimes \bar{f}_i\right)\cdot\left(\sum s_j'\otimes \bar{f}_j'\right)\right) = \varphi\left(\sum s_i s_j'\otimes f_i f_j'\right) = \sum\overline{s_i s_j' f_i f_j'} = \left(\sum\overline{s_i f_i}\right)\left(\sum\overline{s_j' f_j'}\right)$

$= \varphi\left(\sum s_i\otimes f_i\right)\cdot\varphi\left(\sum s_j'\otimes f_j'\right)$, so φ is an isomorphism of S-algebras.