HW 3

$\boxed{10.5}$

2a. Give all the maps names:

$$\begin{tikzcd}
A \arrow[r, "\pi_1"] \arrow[d, "\alpha"'] & B \arrow[r, "\pi_2"] \arrow[d, "\beta"'] & C \arrow[r, "\pi_3"] \arrow[d, "\gamma"] & D \arrow[d, "\delta"] \\
A' \arrow[r, "\sigma_1"'] & B' \arrow[r, "\sigma_2"'] & C' \arrow[r, "\sigma_3"'] & D'
\end{tikzcd}$$

Suppose $s \in C$ s.t. $\gamma(s) = 0$.
Then $\sigma_3(\gamma(s)) = 0$, so $\delta(\pi_3(s)) = \sigma_3(\gamma(s)) = 0$.
Since δ is injective, $\pi_3(s) = 0$.
By exactness, $\exists\, t \in B$ s.t. $\pi_2(t) = s$.
$\sigma_2(\beta(t)) = \gamma(\pi_2(t)) = \gamma(s) = 0$, so $\beta(t) \in \operatorname{Ker} \sigma_2$.
By exactness, $\exists\, u' \in A'$ s.t. $\sigma_1(u') = \beta(t)$.
α is surjective so $\exists\, u \in A$ s.t. $\alpha(u) = u'$.
$\beta(\pi_1(u)) = \sigma_1(\alpha(u)) = \beta(t)$, so $\pi_1(u) = t$, since β is injective.
Thus $s = \pi_2(\pi_1(u))$. But by exactness, $\pi_2 \circ \pi_1 = 0$, so $s = 0$.
Therefore, γ is injective.

b. Let $b' \in B'$ and let $c' = \sigma_2(b')$.
Since γ is surjective $\exists\, c \in C$ s.t. $\gamma(c) = c'$.
$\sigma_3(c') = 0$ by exactness since $c' \in \operatorname{Im} \sigma_2$.
Thus $\delta(\pi_3(c)) = \sigma_3(\gamma(c)) = \sigma_3(c') = 0$.
Since δ is injective, $\pi_3(c) = 0$.
By exactness, $\exists\, b \in B$ s.t. $\pi_2(b) = c$.
Let $b'' = \beta(b)$.
Then $\sigma_2(b'') = \sigma_2(\beta(b)) = \gamma(\pi_2(b)) = \gamma(c) = c'$.
Also, $\sigma_2(b') = c'$, so $\sigma_2(b' - b'') = 0$.
By exactness, $\exists\, a' \in A'$ s.t. $\sigma_1(a') = b' - b''$.
Since α is surjective, $\exists\, a \in A$ s.t. $\alpha(a) = a'$.

Let $s = \pi_1(a)$.

Then ~~[illegible]~~ $\beta(s) = \beta(\pi_1(a)) = \sigma_1(\alpha(a)) = \sigma_1(a') = b' - b''$.

So $\beta(b+s) = \beta(b) + \beta(s) = b'' + (b' - b'') = b'$.

Thus $b' \in \operatorname{Im} \beta$, so β is surjective.

(6) Use parts (3) of propositions 30 and 34. Every R-module is projective iff every short exact sequence of R-modules splits iff every R-module is injective.

(14) a. Applying $\operatorname{Hom}_R(D,-)$ to $0 \to M \to M \oplus N \to N \to 0$ gives the short exact sequence $0 \to \operatorname{Hom}_R(D,M) \to \operatorname{Hom}_R(D,M) \oplus \operatorname{Hom}_R(D,N) \to \operatorname{Hom}_R(D,N) \to 0$.

Conversely, suppose $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ is short exact and that $0 \to \operatorname{Hom}_R(D,L) \xrightarrow{\psi'} \operatorname{Hom}_R(D,M) \xrightarrow{\varphi'} \operatorname{Hom}_R(D,N) \to 0$ is short exact for any R-module D. Then in particular, $\varphi' : \operatorname{Hom}_R(N,M) \to \operatorname{Hom}_R(N,N)$ is surjective. Then $\exists\, \sigma \in \operatorname{Hom}_R(N,M)$ s.t. $\varphi'(\sigma) = I_N$. But $\varphi'(\sigma)$ by definition is just $\varphi \circ \sigma$, so σ gives a splitting for $0 \to L \xrightarrow{\psi} M \underset{\sigma}{\overset{\varphi}{\rightleftarrows}} N \to 0$.

b "$\Leftarrow$" follows as in part a. Suppose conversely that $0 \to \operatorname{Hom}_R(N,D) \xrightarrow{\varphi'} \operatorname{Hom}_R(M,D) \xrightarrow{\psi'} \operatorname{Hom}_R(L,D) \to 0$ for all R-modules D.

In particular, $\psi' : \operatorname{Hom}_R(M,L) \to \operatorname{Hom}_R(L,L)$ is surjective, so $\exists\, \sigma \in \operatorname{Hom}_R(M,L)$ s.t. $\psi'(\sigma) = I_L$. But $\psi'(\sigma) = \sigma \circ \psi$ so σ is a splitting for

$$0 \to L \underset{\sigma}{\overset{\psi}{\rightleftarrows}} M \xrightarrow{\varphi} N \to 0.$$