HW 4

10.5

(20) $R[x]$ is a free R-module with basis $\{1, x, x^2, \dots\}$ and free modules are flat. Done.

(28) a. Follow the hint. Let X be the fiber product (see #27) of φ and φ'. That is, $X = \{(x,x') \in P \times P' \mid \varphi(x) = \varphi'(x')\}$. Define $\pi : X \to P$ by $\pi(x,x') = x$. π is surjective since φ' is surjective. We have a short exact sequence $0 \to \ker\pi \to X \xrightarrow{\pi} P \to 0$. $\ker\pi = \{(0,x') \in X\}$ $= \{(0,x') \in P \times P' \mid \varphi'(x') = 0\} \cong \ker\varphi' \cong K'$. Thus we have an exact sequence $0 \to K' \to X \to P \to 0$, which splits since P is projective. Thus $X \cong K' \oplus P$. Defining $\pi' : X \to P'$ by $\pi'(x,x') = x'$ we get by a similar argument that $X \cong K \oplus P'$. Thus $K' \oplus P \cong K \oplus P'$.

b. The maps ψ and ψ' in the problem statement are useless to me so instead let ψ be the injection $M \hookrightarrow Q$ and ψ' the injection $M \hookrightarrow Q'$. Let X be the fiber sum of (see #27) ψ and ψ'. Then $X = Q \oplus Q' / \{(\psi(m), -\psi'(m)) \mid m \in M\}$. Define $\pi : Q \to X$ by $\pi(x) = \overline{(x,0)}$. π is injective since ψ' is injective. The cokernel $X/\pi(Q)$ is isomorphic to $Q'/\psi'(M) \cong L'$. Thus we have a short exact sequence $0 \to Q \xrightarrow{\pi} X \to L' \to 0$. The sequence splits since Q is injective so $X \cong Q \oplus L'$. Similarly, we can define a map $\pi' : Q' \to X$ and show $X \cong Q' \oplus L$. Thus $Q \oplus L' \cong Q' \oplus L$.

App. II.1

(1) a. If $\varphi : A \to B$ and $\psi : B \to C$ are homomorphisms with $\varphi(N) \subseteq N$ and $\psi(N) \leq N$ then clearly $(\psi \circ \varphi)(N) \leq N$, so we have a law of composition, which is clearly associative. The identity morphism $A \to A$ takes N to N so identity morphisms exist in Nor-N. So we have a category.

b We attempt to define a functor F from Nor-N to Grp with $F(G) = G/N$ for any object G of Nor-N. Given $f \in \mathrm{Hom}_{\text{Nor-}N}(A, B)$ define $F(f) \in \mathrm{Hom}_{Grp}(A/N, B/N)$ by $(F(f))(\bar{a}) = \overline{f(a)}$. This is well-defined since $f(N) \leq N$. We clearly have $F(gf) = F(g)F(f)$ and $F(1_A) = 1_{F(A)}$ So F is a functor.