HW 5

App II Sec. 2

① Define $\mathcal{G}G = G/N$. If $f \in \mathrm{Hom}_{Nor\text{-}N}(G,H)$, define $\mathcal{G}(f) \in \mathrm{Hom}_{Grp}(\mathcal{G}G, \mathcal{G}H)$ by $(\mathcal{G}(f))(\bar{g}) = \overline{f(g)}$. Define a natural transformation η from F to $\mathcal{G}$ by defining $\eta_G : \underset{\substack{\| \\ G}}{FG} \longrightarrow \underset{\substack{\| \\ G/N}}{\mathcal{G}G}$ to be the quotient map for all G in Nor-N. Suppose

$f \in \mathrm{Hom}_{Nor\text{-}N}(G,H)$. The diagram below is clearly commutative so η is a natural transformation.

$$\begin{tikzcd}
G = FG \arrow[r, "\eta_G"] \arrow[d, "f = F(f)"'] & \mathcal{G}G = G/N \arrow[d, "\mathcal{G}(f) = \bar{f}"] \\
H = FH \arrow[r, "\eta_H"] & \mathcal{G}H = H/N
\end{tikzcd}$$

③ Let G be a group. The commutator subgroup G' has the following universal property: For any homomorphism $f: G \to A$ with A an abelian group, there is a unique homomorphism $g: G/G' \to A$ such that the diagram below commutes.

$$\begin{tikzcd}
G \arrow[r] \arrow[dr, "f"'] & G/G' \arrow[d, "g"] \\
& A
\end{tikzcd}$$

Let F be the forgetful functor from Ab to Grp. In the notation of page 917, X is G, $U(X)$ is G/G', and ι is the quotient map $G \to G/G'$. Our commutative diagram shows that $(G/G', \iota)$ is a universal arrow from G to F.

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(2) a. Let's prove the equivalent statement in the parentheses. $x_1, \dots, x_n$ are linearly independent by assumption. Let $y \in M$. By maximality of $x_1, \dots, x_n$, the set $\{x_1, \dots, x_n, y\}$ is linearly dependent. Hence $\exists r, r_1, \dots, r_n \in R$ s.t. $ry = \sum r_i x_i$. Since the x_i are linearly independent we must have $r \neq 0$.

b. Let $x_1, \dots, x_n$ be a basis for N and let $y_1, \dots, y_{n+1} \in M$. By part (a) $\exists r_1, \dots, r_{n+1}$ s.t. $r_i y_i \in N$ $\forall i$. To show $y_1, \dots, y_{n+1}$ are linearly dependent it's equivalent to show $r_1 y_1, \dots, r_{n+1} y_{n+1}$ are linearly dependent. So replacing y_i with $r_i y_i$, we may assume $y_i \in N$ $\forall i$. Since N is free of rank n, the linear dependence of the y_i follows from Prop. 3.