HW 6

$\boxed{12.1}$

(9.) Take $R = \mathbb{Z}$ and $M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \cdots$. Now let N be a finitely generated torsion R-module. Let $x_1, \ldots, x_n$ generate N and for each $i = 1, \ldots, n$, let $0 \neq r_i \in R$ annihilate x_i. Then $0 \neq r_1 \cdots r_n$ annihilates every x_i and hence annihilates N.

(14.) Let $0 \neq m \in M$. Then Rm is a nonzero submodule of M. Assuming M is irreducible, this forces $M = Rm$. So $M \cong R/(a)$ for some $a \in R$. Since M is torsion, $a \neq 0$. Let $p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ be a prime factorization of a. If $a \neq p_1$, then $\overline{p_1^{\alpha_1 - 1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}} \in R/(a)$ generates a proper nonzero submodule of M. Since M is irreducible, we must have $a = p_1$. Conversely, suppose $M = R/(p)$ for some nonzero prime ideal (p). (p) is maximal so $R/(p)$ is a field. It follows that any nonzero element of $R/(p)$ generates it as an $R/(p)$-module, and hence as an R-module. Thus M is irreducible.

(21) By Theorem 5 and #3 in 10.5 we can restrict ourselves to showing R is projective and $R/(a)$ is not when $a \neq 0$ (R is our P.I.D.). R is free hence projective. For $R/(a)$ consider

$$\begin{tikzcd} & R/(a) \arrow[dl, dashed, "\varphi"'] \arrow[d, "\cong"] & \\ R \arrow[r, two heads] & R/(a) \arrow[r] & 0 \end{tikzcd}$$

If φ exists, then $\varphi(\overline{1})$ is annihilated by a since $a\varphi(\overline{1}) = \varphi(\overline{a}) = \varphi(\overline{0}) = 0$. But R has no zero divisors so we must have $\varphi(\overline{1}) = 0$ and then the diagram doesn't commute.