# HW 7

## 12.2

4. Two matrices are similar iff their rational canonical forms are the same iff their lists of invariant factors are the same. The characteristic polynomial $c_A(x)$ is the product of the invariant factors and the minimal polynomial $m_A(x)$ is the largest invariant factor.

For a $3\times 3$ matrix $A$, $c_A(x)$ has degree 3. If $\deg m_A(x) = 1$, then we must have $m_A(x) = x-b$ and $c_A(x) = (x-a)^3$ for some $b$. It follows that the invariant factors are $a_1 = a_2 = a_3 = x-b$.

If $\deg m_A(x) = 2$, it follows that $a_1$ is the linear factor $(x-b)$ s.t. $m_A(x)(x-b) = c_A(x)$ and $a_2 = m_A(x)$.

If $\deg m_A(x) = 3$, the we can only have one invariant factor: $a_1 = m_A(x)$.

In all 3 cases, $c_A(x)$ and $m_A(x)$ uniquely determine the invariant factors, so we're done.

For the $4\times 4$ counterexample, consider the lists of invariant factors

1) $a_1 = a_2 = x$, $a_3 = x^2$

2) $a_1 = a_2 = x^2$

In both cases, the characteristic polynomial is $x^4$ and the minimal polynomial is $x^2$. But the invariant factors are different so the corresponding matrices

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

in rational canonical form cannot be similar.

(14) The possible lists of invariant factors are:

1) $a_1 = x^2(x^2+1)^2 = x^6+2x^4+x^2 \Longrightarrow \begin{bmatrix} 0 & & & & & 0 \\ 1 & 0 & & & & 0 \\ & 1 & 0 & & & -1 \\ & & 1 & 0 & & 0 \\ & & & 1 & 0 & -2 \\ & & & & 1 & 0 \end{bmatrix}$

2) $a_1 = x \quad a_2 = x(x^2+1)^2 = x^5+2x^3+x \Rightarrow \left[\begin{array}{c|ccccc} 0 & & & & & \\ \hline & 0 & & & & 0 \\ & 1 & 0 & & & -1 \\ & & 1 & 0 & & 0 \\ & & & 1 & 0 & -2 \\ & & & & 1 & 0 \end{array}\right]$

3) $a_1 = x^2+1 \quad a_2 = x^2(x^2+1) = x^4+x^2 \Rightarrow \left[\begin{array}{cc|cccc} 0 & -1 & & & & \\ 1 & 0 & & & & \\ \hline & & 0 & & & 0 \\ & & 1 & 0 & & 0 \\ & & & 1 & 0 & -1 \\ & & & & 1 & 0 \end{array}\right]$

4) $a_1 = a_2 = x(x^2+1) = x^3+x \Rightarrow \left[\begin{array}{ccc|ccc} 0 & & 0 & & & \\ 1 & 0 & -1 & & & \\ & 1 & 0 & & & \\ \hline & & & 0 & & 0 \\ & & & 1 & 0 & -1 \\ & & & & 1 & 0 \end{array}\right]$

$\boxed{13.1}$

(8) Use the rational roots test to show $x^5-ax-1$ has a linear factor iff $a$ is 0 or 2. Suppose $x^5-ax-1 = (x^3+b_1x^2+b_2x+b_3)(x^2+c_1x+c_2)$. Comparing coefficients, we get $c_1 = -b_1$, $c_2 = \frac{-1}{b_3}$, and $b_2 = b_1^2 + \frac{1}{b_3}$.

So, $x^5-ax-1 = (x^3+b_1x^2+(b_1^2+\frac{1}{b_3})x+b_3)(x^2-b_1x-\frac{1}{b_3})$. Comparing some more, we get $\frac{-b_1}{b_3} - b_1(b_1^2+\frac{1}{b_3}) + b_3 = 0$ and $\frac{-1}{b_3}(b_1^2+\frac{1}{b_3}) - b_1b_3 = -a$.

The coefficients of the polynomials are integers, so $b_3 = \pm 1$. Checking each case (left to reader), we get $a=-1$, $b_1 = 1$, $b_3 = -1$.

$\boxed{13.2}$

(3) $x = 1+i$

$x^2 = 2i$

So, $x^2 - 2x + 2 = 0$. $x^2 - 2x + 2$ is irreducible over $\mathbb{Q}$ since it has no roots in $\mathbb{Q}$ (also by Eisenstein), so it's the minimal polynomial of $1+i$.

(14) Suppose $F(\alpha) \neq F(\alpha^2)$. Then $[F(\alpha) : F(\alpha^2)] = 2$ since $x^2 - \alpha^2$ is the minimal polynomial of $\alpha$ over $F(\alpha^2)$. But then $[F(\alpha) : F] = [F(\alpha) : F(\alpha^2)][F(\alpha^2) : F]$ is even, contradiction.